Q: Given a function $f$, what are the maximum and minimum values of $f(x)$?

**Defn:** Let $f$ be a function on an interval $I$ and let $c \in I$.

Then $f(c)$ is:

- **absolute min:** if $f(c) \leq f(x)$ for all $x \in I$
- **absolute max:** if $f(c) \geq f(x)$ for all $x \in I$

**Remark:** absolute extrema do not always exist. (See page 200-201.)

**Theorem (Existence of absolute extrema):**

A continuous function $f$ on a closed and bounded interval $[a, b]$ takes on an absolute min and max value.
While absolute min and max may not exist for the function's entire domain, we can still restrict the domain and look for local extrema:

**Defn:** \( f(c) \) is a:

- **local min**: if \( f(c) \) is a minimum value on some open interval
- **local max**: if \( f(c) \) is a maximum value on some open interval

**Eg.**

\[ y = f(x) \]

- **absolute max**
- **local max**
- **local min**
- **absolute min**

Q: What happens to \( f'(x) \) at these points?
**Defn**: A number \( c \in \text{dom}(f) \) is called a critical point if either:

1. \( f'(c) = 0 \), or

2. \( f'(c) \) DNE

**Thm** (Fermat's theorem for local extrema)

\[ f(c) \text{ local min or max} \quad \Rightarrow \quad c \text{ is a critical point} \]

**Remark**: Caution! The other direction is not true in general.

*E.g.*, \( f'(a) = 0 \), but \( x=a \) not local min/max
Examples (§ 4.2)

10. \( f(x) = \frac{x^2}{x^2 - 4x + 8} \); find all critical points.

\[
f'(x) = \frac{2x(x^2 - 4x + 8) - x^2(2x - 4)}{(x^2 - 4x + 8)^2} \]
\[
= \frac{2x^3 - 8x^2 + 16x - 2x^3 + 4x^2}{(x^2 - 4x + 8)^2} \]
\[
= \frac{16x - 8x^2}{(x^2 - 4x + 8)^2} \]

Now, note that \( x^2 - 4x + 8 \neq 0 \) for all \( x \) (check the discriminant! \( (-4)^2 - 4(1)(8) < 0 \)) and see that \( 0 = 16x - 8x^2 = x(16 - 8x) \)

\[\Rightarrow x = 0 \text{ or } 16 - 8x = 0 \]

\[\Rightarrow x = 0 \text{ or } x = 2 \]

Hence the derivative is defined everywhere and \( = 0 \) at \( x = 0, 2 \).
\[ f(t) = 4t - \sqrt{t^2+1} \]

Find all critical points.

\[ f'(t) = 4 - \frac{1}{2}(t^2+1)^{-\frac{1}{2}}(2t) \]

\[ = 4 - \frac{t}{\sqrt{t^2+1}} \]

First, note that \( t^2+1 > 0 \) for all \( t \), hence \( f' \) is defined everywhere.

Now suppose \( 0 = f'(t) = 4 - \frac{t}{\sqrt{t^2+1}} \)

\[ \Rightarrow 4\sqrt{t^2+1} = t \]

\[ \Rightarrow 16(t^2+1) = t^2 \]

\[ \Rightarrow 16t^2 + 16 = t^2 \Rightarrow 16 = -15t^2 \]

\[ \Rightarrow \frac{-16}{15} = t^2 \]

Hence there are no such \( t \), and hence \( f \) has no critical points.
(14) \( f(x) = x + \left| 2x + 1 \right| \); find all critical points.

First, rewrite \( f \) as a piecewise function:

\[
f(x) = \begin{cases} 
  x + 2x + 1, & 2x + 1 \geq 0 \\
  x - (2x + 1), & 2x + 1 < 0
\end{cases}
\]

\[
= \begin{cases} 
  3x + 1, & x \geq -\frac{1}{2} \\
  -x - 1, & x < -\frac{1}{2}
\end{cases}
\]

Now consider the derivative for \( x \neq -\frac{1}{2} \):

\[
f'(x) = \begin{cases} 
  3, & x > -\frac{1}{2} \\
  -1, & x < -\frac{1}{2}
\end{cases}
\]

What about at \( x = -\frac{1}{2} \)? We must use the limit definition:

\[
f'(-\frac{1}{2}) = \lim_{t \to \frac{1}{2}} \frac{f(t) - f(-\frac{1}{2})}{t - \frac{1}{2}} = \lim_{t \to \frac{1}{2}} \frac{f(t) + \frac{1}{2}}{t + \frac{1}{2}}
\]
Now consider left and right limits:

\[
\lim_{{t \to (\frac{1}{2})^-}} \frac{f(t) + \frac{1}{2}}{t + \frac{1}{2}} = \lim_{{t \to (\frac{1}{2})^-}} \frac{-2 - 1 + \frac{1}{2}}{t + \frac{1}{2}} \\
= \lim_{{t \to (\frac{1}{2})^-}} \frac{- (t + \frac{1}{2})}{t + \frac{1}{2}} = -1
\]

and

\[
\lim_{{t \to (\frac{1}{2})^+}} \frac{f(t) + \frac{1}{2}}{t + \frac{1}{2}} = \lim_{{t \to (\frac{1}{2})^+}} \frac{3t + 1 + \frac{1}{2}}{t + \frac{1}{2}} \\
= \lim_{{t \to (\frac{1}{2})^+}} \frac{3(t + \frac{1}{2})}{t + \frac{1}{2}} = 3
\]

Hence left limit \( \neq \) right limit, so

\[
\frac{d}{dx} f(\frac{1}{2}) \text{ does not exist}
\]

So \( x = \frac{-1}{2} \) is a critical point.
15) \( g(\theta) = \sin^2 \theta \); find critical points

\[ g'(\theta) = 2\sin \theta \cos \theta \]

\[ \Rightarrow \quad 0 = 2\sin \theta \cos \theta \]

\[ \Rightarrow \quad 0 = \sin \theta \cos \theta \quad \Rightarrow \quad \sin \theta = 0 \quad \text{or} \quad \cos \theta = 0 \]

The zeros of sine are:

\[ \{k\pi \mid k \in \mathbb{Z}\} \]

The zeros of cosine are:

\[ \{\frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z}\} \]

So the set of critical points is:

\[ \{k\pi \mid k \in \mathbb{Z}\} \cup \{\frac{(2k+1)\pi}{2} \mid k \in \mathbb{Z}\} \]

(Notice: there are infinitely many critical points here!)
24. Compute critical points of

\[ h(t) = (t^2 - 1)^{\frac{1}{3}} \] and compare with graph. (Then find extreme values of \( h \) on \([0,1]\) and \([0,2]\))

\[ h'(t) = \frac{1}{3} (t^2 - 1)^{-\frac{2}{3}} (2t) = \frac{2t}{3 (t^2 - 1)^{\frac{2}{3}}} \]

Now, suppose

\[ 3(t^2 - 1)^{\frac{2}{3}} = 0 \quad \Rightarrow \quad 27(t^2 - 1)^2 = 0 \]

\[ \Rightarrow \quad t^2 - 1 = 0 \quad \Rightarrow \quad t = \pm 1 \]

So \( h' \) is not defined at \( t = \pm 1 \)

Now suppose

\[ 2t = 0 \quad \Rightarrow \quad t = 0 \]

So \( h'(t) = 0 \) at \( t = 0 \)
So the critical points are:

\[ \{-1, 0, 1\} \]

(Note the absolute min at \( t=0 \))

\( \Rightarrow \) The extreme values on \([0, 1]\)
occurs at \( t=0 \) (minimum) and \( t=1 \) (maximum)

\( \Rightarrow \) The extreme value on \([0, 2]\)
occurs at \( t=0 \) (min) and \( t=2 \) (max)

\[ \downarrow \]

Why do we know these?

\( \Rightarrow \) Think about the values of the derivative!
(\( \text{can you use the IVT?} \))
\[ f(\theta) = \cos \theta + \sin \theta, \quad [0, 2\pi] \]

\[ f'(\theta) = -\sin \theta + \cos \theta \]
so \[ f'(\theta) = 0 \iff \sin \theta = \cos \theta \]

which happens only at \( \theta = \frac{\pi}{4} \) and \( \theta = \frac{5\pi}{4} \).

Extreme values may also occur at the end points, \( \theta = 0 \) and \( \theta = 2\pi \).

Now we check:

\[ f(0) = 1 \]
\[ f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = \sqrt{2} \]
\[ f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} + -\frac{1}{\sqrt{2}} = -\sqrt{2} \]
\[ f(2\pi) = 1 \]

Therefore, the \underline{absolute max} of \( f \) on \([0, 2\pi]\)

is \( \sqrt{2} \)

and the \underline{absolute min} of \( f \)

on \([0, 2\pi]\) is \( -\sqrt{2} \)
\[ f(x) = x^3 - 24 \ln(x), \ [\frac{1}{2}, 3] \]

\[ f'(x) = 3x^2 - \frac{24}{x} \]

- \[ f'(x) \text{ is undefined for } x = 0 \]
- \[ f'(x) = 0 \text{ for } 3x^2 = \frac{24}{x} \implies 3x^3 = 24 \implies x^3 = 8 \implies x = 2 \]

Now we check at the critical points \( \{2\} \) and the end points \( \{\frac{1}{2}, 3\} \)

\[ f\left(\frac{1}{2}\right) = \frac{1}{8} - 24 \ln\left(\frac{1}{2}\right) = \frac{1}{8} + 24 \ln(2) \]

\[ f(2) = 8 - 24 \ln(2) \text{ max} \]

\[ f(3) = 2.7 - 24 \ln(3) \text{ min} \]
Procedure to find extrema of continuous function $f$ on a closed interval $[a, b]$

1. Find the critical points of $f$ in the interval $[a, b]$.

2. Compute $f(x)$ for the critical points $x$ and the endpoints $x = a$ and $x = b$.

3. The largest (resp. smallest) value among those in step 2 is the absolute max (resp. min.) of $f(x)$ on $[a, b]$. 
Consequence of optimization procedure on a closed interval:

Rolle's theorem:

Thm. (Rolle) \( f \) continuous on \([a, b]\) and differentiable on \((a, b)\).

If \( f(a) = f(b) \), then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

Example:

72) Prove that 4 is the largest real root of \( f(x) = x^4 - 8x^2 - 128 \).

(Remark: this is an example of proof by contradiction.)

Suppose that there exists some \( a > 4 \) such that \( f(a) = 0 \). Now, since we've given that \( f(4) = 0 \), by Rolle's theorem there exists some \( c \in (4, a) \) such that \( f'(c) = 0 \).

cont'd
However, see that:
\[ f'(x) = 4x^3 - 16x \]
\[ = 4x(x^2 - 4) = 4x(x - 2)(x + 2) \]
hence the only zeros of \( f' \) are at 0, 2, and -2.

Therefore, \( f' \) cannot have a zero in the interval \((4, a)\) for any \( a \), hence we've arrived at a contradiction:
\[ f'(c) \neq 0 \text{ for all } c \in (4, a). \]

And hence \( f \) must not have a real root greater than 4.
Mean Value Theorem

"Fundamental theorem of differentiable functions"

Thm. (MVT) \( f \) continuous on \([a, b]\) 
\( f \) differentiable on \((a, b)\)

Then there exists \( c \in (a, b) \) such that

\[
  f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Remark: If \( f(a) = f(b) \) then 
(Rolle's theorem \( \Rightarrow \) MVT)

Idea: The secant line for the points \((a, f(a))\) and \((b, f(b))\) has the same slope as some tangent line to the graph \( y = f(x) \) for some point between \( a \) and \( b \) (this point is \( c \) )
(9) \( y = x^2 \)

Find a point \( x = c \) in \([0,1]\) satisfying the conclusion of the MVT and graph the resulting tangent and secant lines:

We want \( c \in (0,1) \) such that:

\[
y'(c) = \frac{y(1) - y(0)}{1 - 0} = 1
\]

hence \( 2c = y'(c) = 1 \) \( \Rightarrow \) \( c = \frac{1}{2} \)

---

![Graph of \( y = x^2 \) with tangent and secant lines at \( c = \frac{1}{2} \)]

notre the lines are parallel & same slope!
\[ y = e^x \text{ on } [0, 1] \; \text{; same question as before:} \]

So we want \( c \in (0, 1) \) satisfying

\[ y'(c) = \frac{y(1) - y(0)}{1 - 0} = \frac{e - 1}{1} = e - 1 \]

hence \( e^c = e - 1 \Rightarrow c = \ln(e - 1) \)
Sam made two statements that Deb found dubious:

(a) "The average velocity for my trip was 70 mph, but at no point did my speedometer read 70 mph."

(b) "A policeman clocked me at 70 mph, but my speedometer never read 65 mph."

Prove these statements false with theorems

(a) Let \( s(t) \) be the position function for Sam and let \([0,a]\) be the time interval of his trip. Then the statement is equivalent to:

\[
\text{for any } c \in (0,a), \quad s'(c) \neq 70 = \frac{s(a) - s(0)}{a - 0}
\]

"speedometer never read 70"

Which contradicts MVT since \( s(t) \) is continuous and differentiable.
(b) The statement is equivalent to:

"\[ s'(b) = 70 \text{ for some } b \in (0, a), \]
but \[ s'(c) \neq 65 \text{ for all } c \in [0, a] \]

Now, we take \( s'(a) = 0 \) since we can assume Sam started his trip from a stop.

Therefore \( s'(0) = 0 \) and \( s'(b) = 70 \), but there is no \( c \in (0, b) \) such that \( s'(c) = 65 \).

This contradicts the IVT since \( s' \) cannot "skip" 65.
Corollary (to MVT)

If \( f \) is differentiable and \( f'(x) = 0 \) for all \( x \in (a, b) \), then \( f \) is constant on \( (a, b) \), i.e.

\[ f(x) = C \quad \text{for some } C \in \mathbb{R} \]

Monotone functions + 1st derivative test:

A function \( f \) is:

- Increasing on \( (a, b) \) if \( f(x) < f(y) \) for all \( x, y \in (a, b) \) with \( x < y \)

- Decreasing on \( (a, b) \) if \( f(x) > f(y) \) for all \( x, y \in (a, b) \) with \( x < y \)

Theorem (Sign of the derivative)

\( f \) differentiable on \( (a, b) \)

1. \( f'(x) > 0 \) for all \( x \in (a, b) \) \( \Rightarrow \) \( f \) increasing on \( (a, b) \)
2. \( f'(x) < 0 \) for all \( x \in (a, b) \) \( \Rightarrow \) \( f \) decreasing on \( (a, b) \)
we can use the preceding theorem to determine when a critical point gives a local max or min.

**Theorem** (First derivative test)

Let $f$ be continuous and differentiable on an interval containing critical point $c$, then for $a < c < b$,

1. If $f'(a) > 0$ and $f'(b) < 0$
   Then $f(c)$ is **local max**

2. If $f'(a) < 0$ and $f'(b) > 0$
   Then $f(c)$ is **local min**

That is:

1. $f'(x)$ changes $+$ to $-$ at $c$ \( \Rightarrow f(c) \) local max

2. $f'(x)$ changes $-$ to $+$ at $c$ \( \Rightarrow f(c) \) local min
Examples (§ 4.3)

38) \[ y = x^{5/2} - x^2 \quad (x > 0) \]

Find the critical points, the intervals of increase/decrease, and the local extrema.

\[ y'(x) = \frac{5}{2} x^{3/2} - 2x = x \left( \frac{5}{2} \sqrt{x} - 2 \right) \]

Since \( x > 0 \), \( y'(x) \) is defined everywhere.

Now we'll find the zeros:

\[ 0 = y'(x) = x \left( \frac{5}{2} \sqrt{x} - 2 \right) \]

hence \( x = 0 \) or \( \frac{5}{2} \sqrt{x} = 2 \)

\[ \Rightarrow \frac{5}{2} \sqrt{x} = 2 \]

\[ \Rightarrow \frac{25}{4} x = 4 \quad \Rightarrow x = \frac{16}{25} \]

Now, \( y'(x) \) is continuous and has only one zero in \( (0, \infty) \) given by \( \frac{16}{25} = x \).

Hence since for \( \left( \frac{1}{25} < \frac{16}{25} \right) \)

\[ y'(\frac{16}{25}) = \frac{1}{25} \left( \frac{5}{2} \sqrt{\frac{16}{25}} - 2 \right) = \frac{1}{50} - \frac{2}{25} \]

\[ \text{critical point} \]
\[ \frac{1}{50} - \frac{4}{50} = -\frac{3}{50} < 0 \quad \text{neg} \]
and for \( \frac{16}{25} < 1 \)
\[ y'(1) = (1)(\frac{5}{2} - 2) = \frac{1}{2} > 0 \quad \text{pos} \]
Hence by IVT \( y' \) only changes signs once, at \( x = \frac{16}{25} \) and from negative to positive.

Thus \( y \) is decreasing on \( (0, \frac{16}{25}) \)
and increasing on \( (\frac{16}{25}, \infty) \)

Finally, we can check the local extrema; the only possibility is at \( x = \frac{16}{25} \), and we've just shown that \( y' \) changes from neg. to pos. there; hence

\[ x = \frac{16}{25} \] gives a local minimum
\( y = \frac{2x+1}{x^2+1} \); Find same info as before.

\[
y'(x) = \frac{2(x^2+1) - (2x+1)(2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2} = \frac{2(1-x-x^2)}{(x^2+1)^2}
\]

Clearly \( x^2+1 \geq 1 > 0 \), hence \( y'(x) \) is defined everywhere.

If \( y'(x) = 0 \) then \( 2(1-x-x^2) = 0 \), hence \( y' \) is zero at

\[
x = \frac{-(1) \pm \sqrt{(1)^2 - 4(-1)(1)}}{2(-1)} = \frac{1 \pm \sqrt{1+4}}{-2} = \frac{-(1 \pm \sqrt{5})}{2}
\]

So the critical points are

\[
\text{cp}(y) = \left\{ \frac{-(1+\sqrt{5})}{2}, \frac{-(1-\sqrt{5})}{2} \right\}
\]
Now consider the sign changes at the critical points:

<table>
<thead>
<tr>
<th>Sign</th>
<th>$x$</th>
<th>$y'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-</td>
<td>-2</td>
<td>$\frac{-2}{25}$</td>
</tr>
<tr>
<td>(CP)</td>
<td>$-(1+\sqrt{5})$</td>
<td>0</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(CP)</td>
<td>$-(1-\sqrt{5})$</td>
<td>0</td>
</tr>
<tr>
<td>-</td>
<td>2</td>
<td>$\frac{-2}{5}$</td>
</tr>
</tbody>
</table>

Hence by an theorem + IVT,
y is decreasing on $(-\infty, \frac{-(1+\sqrt{5})}{2})$,
increasing on $\left(\frac{-(1+\sqrt{5})}{2}, \frac{-(1-\sqrt{5})}{2}\right)$,
and decreasing on $\left(\frac{-(1-\sqrt{5})}{2}, \infty\right)$.
The chart of values also shows that

\[ y \text{ has local min at } x = \frac{-(14-\sqrt{5})}{2} \]

and \( y \) has local max at \( x = \frac{-(1+\sqrt{5})}{2} \)

\[ y = (x^2-2x)e^x; \text{ find same info as before} \]

\[ y'(x) = (2x-2)e^x + (x^2-2x)e^x \]

\[ = (x^2-2)e^x; \text{ clearly } y' \text{ is defined everywhere} \]

Since \( e^x \neq 0 \) for all \( x \in \mathbb{R} \), we need only find the zeros of \( (x^2-2)e^x \)

\[ (x^2-2) = (x-\sqrt{2})(x+\sqrt{2}) \]

hence the critical points are

\[ \text{cp}(y) = \left\{ \sqrt{2}, -\sqrt{2} \right\} \]
Now consider the sign changes:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y'(x)$</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>$\frac{2}{e^2}$</td>
<td>+</td>
</tr>
<tr>
<td>$-\sqrt{2}$</td>
<td>0</td>
<td>(CP)</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
<td>-</td>
</tr>
<tr>
<td>$\sqrt{2}$</td>
<td>0</td>
<td>(CP)</td>
</tr>
<tr>
<td>2</td>
<td>$2e^2$</td>
<td>+</td>
</tr>
</tbody>
</table>

Hence by our theorems and IVT, we have:

$y$ is increasing on $(-\infty, -\sqrt{2})$, decreasing on $(-\sqrt{2}, \sqrt{2})$, and increasing on $(\sqrt{2}, \infty)$.

And $y$ has local max at $x = -\sqrt{2}$ and local min at $x = \sqrt{2}$. 
Concavity + 2nd derivative test:

The concavity of a graph is the property of being "cup-like" or "cap-like."

- **Concave up:** "cup-like"
  
  ![Concave up graph](image)

- **Concave down:** "cap-like"
  
  ![Concave down graph](image)

**Definition:** for $f$ differentiable on $(a,b)$ we have:

- $f$ is concave up on $(a,b)$ if $f'$ is increasing
- $f$ is concave down on $(a,b)$ if $f'$ is decreasing
The preceding definition allows us to apply the First derivative test to get:

**[Thm] (Test for concavity)**

Suppose \( f' \) is differentiable on \((a, b)\)

Then:
1. \( f''(x) > 0 \) for \( x \in (a, b) \)
   \[ \Rightarrow f \text{ is concave up on } (a, b) \]
2. \( f''(x) < 0 \) for \( x \in (a, b) \)
   \[ \Rightarrow f \text{ is concave down on } (a, b) \]

and

**[Thm] (Test for inflection)**

If \( f''(c) = 0 \) or \( f''(c) \) DNE

and the sign of \( f''(x) \) changes at \( c \)

Then \( f \) has an inflection point at \( x = c \)

(change of concavity)
Examples (§4.4)

8) \( y = t + \sin^2 t \), \([0, \pi]\)

Find the intervals of concavity and the inflection points

\[
y'(t) = 1 + 2 \sin(t) \cos(t)
\]

\[
y''(t) = 2 (\cos(t) \cos(t) + \sin(t)(-\sin(t)))
= 2 \cos^2(t) - 2 \sin^2(t)
\]

Now, this is defined everywhere, and if \( y''(t) = 0 \) then:

\[2 \cos^2(t) = 2 \sin^2(t)\]

\[\Rightarrow \cos(t) = \pm \sin(t)\]

Clearly this only occurs at \( t = \frac{\pi}{4} \) and \( t = \frac{3\pi}{4} \);

see the graph:

\[y = \sin(t)\]
\[y = -\sin(t)\]
\[y = \cos(t)\]
So now we consider the sign changes for $f''(x)$:

<table>
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<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>$+$</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>0</td>
<td>zero</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$-2$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\frac{3\pi}{4}$</td>
<td>0</td>
<td>zero</td>
</tr>
<tr>
<td>$\pi$</td>
<td>2</td>
<td>$+$</td>
</tr>
</tbody>
</table>

hence we have:

- $y$ is concave up on $[0, \frac{\pi}{4})$,
- concave down on $(\frac{\pi}{4}, \frac{3\pi}{4})$,
- and concave up on $[\frac{3\pi}{4}, \pi]$.

and $y$ has inflection points at both
- $x = \frac{\pi}{4}$ from concave up to down
- and $x = \frac{3\pi}{4}$ from concave down to up.
\[ y = \frac{x}{x^2 + 9} \] ; same question

\[
y'(x) = \frac{(1)(x^2+9) - (x)(2x)}{(x^2+9)^2}
\]

\[
= \frac{x^2 + 9 - 2x^2}{(x^2+9)^2} = \frac{9 - x^2}{(x^2+9)^2}
\]

\[
y''(x) = \frac{(-2x)(x^2+9)^2 - (9-x^2)(2(x^2+9)(2x))}{(x^2+9)^4}
\]

\[
= \frac{-2x(x^2+9)^2 - 4x(9-x^2)(x^2+9)}{(x^2+9)^4}
\]

\[
= \frac{-2x(x^2+9) - 4x(9-x^2)}{(x^2+9)^3}
\]

\[
= \frac{-2x^3 - 18x - 36x + 4x^3}{(x^2+9)^3}
\]

\[
= \frac{2x^3 - 54x}{(x^2+9)^3}
\]
Now, clearly \((x^2+9)^3 > 0\), so we only need to find the zeros of \(y''(x)\).

This is where \(2x^3 - 54x = 0\)

\[
\Rightarrow 2x(x^2 - 27) = 0
\]

\[
\Rightarrow 2x(x - \sqrt{27})(x + \sqrt{27}) = 0
\]

hence \(y''\) is zero only at the points \(\{0, \sqrt{27}, -\sqrt{27}\}\).

Now consider the sign changes:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y''(x))</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9</td>
<td>(-18(54)) / 90³</td>
<td>-</td>
</tr>
<tr>
<td>-(\sqrt{27})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>(\frac{1}{3.18})</td>
<td>+</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-3</td>
<td>(\frac{1}{3.18})</td>
<td>-</td>
</tr>
<tr>
<td>(\sqrt{27})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>(\frac{18(54)}{90³})</td>
<td>+</td>
</tr>
</tbody>
</table>
hence now we see:

\[ y' = \begin{align*}
\text{concave down on } (-\infty, -\sqrt{27}) \\
\text{concave up on } (-\sqrt{27}, 0) \\
\text{concave down on } (0, \sqrt{27}) \\
\text{concave up on } (\sqrt{27}, \infty)
\end{align*} \]

and clearly each of \(-\sqrt{27}, 0, \sqrt{27}\) are all inflection points changing from concave down to up:

\[ \begin{align*}
-\uparrow & \implies \text{up to down} \\
-\downarrow & \implies \text{down to up}
\end{align*} \]

respectively
Finally, we can also use the 2nd derivative to test for extrema:

**Thm**: (2nd derivative test)

Let \( c \) be a critical point for \( f \).

If \( f''(c) \) exists then:

1. \( f''(c) > 0 \Rightarrow f(c) \text{ local min} \)
2. \( f''(c) < 0 \Rightarrow f(c) \text{ local max} \)
3. \( f''(c) = 0 \Rightarrow \text{INCONCLUSIVE} \)

**Examples cont’d (5.4.4)**

\[ f(x) = x^2(x-4) \]

Find: intervals of concavity; points of inflection; critical points; local extrema

First, see that \( f(x) = x^3 - 4x^2 \)

So now: \( f'(x) = 3x^2 - 8x \)

\( f''(x) = 6x - 8 \)
So now the critical points are where \( 0 = f'(x) = 3x^2 - 8x \)
\[ = x(3x-8) \]

Hence either
\[ x = 0 \text{ or } (3x-8) = 0 \]
\[ \Rightarrow x = \frac{8}{3} \]

So:
\[ \text{cp}(f) = \left\{ 0, \frac{8}{3} \right\} \]  

And the zeros of \( f''(x) \) are:
\[ 0 = f''(x) = 6x - 8 \]
\[ \text{are } x = \frac{8}{6} \]

Now we'll consider the following charts to determine the remaining information:
\[
\begin{array}{|c|c|c|c|}
\hline
x & f'(x) & f''(x) & 2\text{nd deriv test} \\
\hline
0 & 0 & -8 & \text{local max} \\
\hline
\frac{8}{3} & 0 & 8 & \text{local min} \\
\hline
\end{array}
\]

Hence \( f \) has an inflection point at \( x = \frac{8}{3} \).

and local min at \( x = \frac{8}{3} \).

\[
\begin{array}{|c|c|c|c|}
\hline
x & f''(x) & \text{sign} & \text{concavity} \\
\hline
0 & -8 & - & \text{Down} \\
\hline
\frac{8}{3} & 0 & 0 & \text{Inflection point} \\
\hline
2 & 4 & + & \text{Up} \\
\hline
\end{array}
\]

Hence \( f \) is concave down on \( (-\infty, \frac{8}{3}) \)
and concave up on \( (\frac{8}{3}, \infty) \).

\( f \) has an inflection point at \( x = \frac{8}{3} \) from concave down to up.
\[ f(x) = \frac{1}{x^4 + 1} \]

And same info

\[ f'(x) = -(x^4 + 1)^{-2} \cdot (4x^3) = \frac{-4x^3}{(x^4 + 1)^2} \]

\[ f''(x) = \frac{(-12x^2)(x^4 + 1)^2 - (-4x^3)(2(x^4 + 1)(4x^3))}{(x^4 + 1)^4} \]

\[ = \frac{-12x^2(x^4 + 1)^2 + 32x^6(x^4 + 1)}{(x^4 + 1)^4} \]

\[ = \frac{-12x^2(x^4 + 1) + 32x^6}{(x^4 + 1)^3} = \frac{-12x^6 - 12x^2 + 32x^6}{(x^4 + 1)^3} \]

\[ = \frac{20x^6 - 12x^2}{(x^4 + 1)^3} \]

The critical points are where \( f'(x) = 0 \) (it is defined everywhere):

\[ 0 = -4x^3 \quad \Rightarrow \quad x = 0 \]

So \( \text{cp}(f) = \{ 0, 1 \} \)
and the zeros of the second derivative are

\[ 20x^6 - 12x^2 = 0 \]

\[ \iff 4x^2(5x^4 - 3) = 0 \]

so \( x = 0 \) or \( 5x^4 - 3 = 0 \)

\[ \Rightarrow x^4 = \frac{3}{5} \]

\[ \Rightarrow x = \pm \sqrt[4]{\frac{3}{5}} \]

so we have potential inflection points at \( \left\{-\left(\frac{3}{5}\right)^{\frac{1}{4}}, 0, \left(\frac{3}{5}\right)^{\frac{1}{4}}\right\} \)

Now consider the charts:
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
<th>Second deriv test</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Inconclusive</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>x</td>
<td>x</td>
</tr>
</tbody>
</table>

The second deriv test is inconclusive, but by the sign change of $f'(x)$ at $x=0$ tells us that $f$ has a local max at $x=0$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>4</td>
<td>+</td>
</tr>
<tr>
<td>$-\left(\frac{2}{3}\right)^{\frac{1}{4}}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$-\frac{3}{2}$</td>
<td>$\frac{-20}{2^5} - \frac{12}{2^2} &lt; 0$</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{20}{2^5} - \frac{12}{2^2} &lt; 0$</td>
<td>-</td>
</tr>
<tr>
<td>$\left(\frac{3}{5}\right)^{\frac{1}{4}}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>+</td>
</tr>
</tbody>
</table>

hence $f$ is concave ↑ on $(-\infty, -\left(\frac{2}{3}\right)^{\frac{1}{4}})$, concave ↓ on $(-\left(\frac{2}{3}\right)^{\frac{1}{4}}, \left(\frac{2}{5}\right)^{\frac{1}{4}})$, concave ↑ on $\left(\frac{2}{5}\right)^{\frac{1}{4}}, \infty)$ and has inflection points at $x = -\left(\frac{2}{3}\right)^{\frac{1}{4}}$ and $x = \left(\frac{2}{5}\right)^{\frac{1}{4}}$. 
(56) \( y = \ln(x^2 + 2x + 5) \); same as before

\[
y'(x) = \frac{1}{x^2 + 2x + 5} (2x + 2) = \frac{2x + 2}{x^2 + 2x + 5}
\]

\[
y''(x) = \frac{(2)(x^2 + 2x + 5) - (2x+2)(2x+2)}{(x^2 + 2x + 5)^2}
\]

\[
= \frac{2x^2 + 4x + 10 - (4x^2 + 8x + 4)}{(x^2 + 2x + 5)^2}
\]

\[
= \frac{-2x^2 - 4x + 6}{(x^2 + 2x + 5)^2} = \frac{-2(x^2 + 2x - 3)}{(x^2 + 2x + 5)^2}
\]

Now, note that 0 = \( x^2 + 2x + 5 \)

at \( x = \frac{-2 \pm \sqrt{4 - 40(5)}}{2(1)} = \frac{-2 \pm \sqrt{-16}}{2} \),

hence there are no real roots and both \( y' \) and \( y'' \) are defined everywhere.
So now the critical points are where \( y'(x) = 0 \)

\[ \Rightarrow 2x + 2 = 0 \quad \iff \quad x = -1 \]

And the possible inflection are where \( y''(x) = 0 \)

\[ \Rightarrow x^2 + 2x - 3 = 0 \]

\[ \Rightarrow x = \frac{-2 \pm \sqrt{4 - 4(1)(-3)}}{2(1)} \]

\[ = \frac{-2 \pm \sqrt{4 + 12}}{2} \]

\[ = \frac{-2 \pm 4}{2} = -1 \pm 2 \]

\[ = \boxed{-3 \text{ or } 1} \]

Now we'll consider the following charts:
<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
<th>Second deriv test</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>$\frac{8}{(4)^2}$</td>
<td>local minimum</td>
</tr>
</tbody>
</table>

So $f$ has a local min at $x = -1$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>-2(16-11) &lt; 0</td>
<td>-</td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>-2(-3) / 5^2 &gt; 0</td>
<td>+</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-2(5) / + &lt; 0</td>
<td>-</td>
</tr>
</tbody>
</table>

Hence $f$ is concave ↓ on $(-\infty, -3)$
Concave ↑ on $(-3, 1)$
Concave ↓ on $(1, \infty)$

and has inflection points at both $x = -3$ and $x = 1$. 