1. **GRAPHING FUNCTIONS**

We can use all of the tools we've collected to create accurate sketches of the graphs of functions.

Follow the following procedure:

1. Find the domain of $f$ and the zeros of $f$
2. Find the domain of $f'$ and the zeros of $f'$
   (i.e., find the critical points)
3. Determine the intervals where $f$ is increasing or decreasing using the 1st. deriv. test on the sign of $f'$.
4. Determine the local extrema
5. Find the domain of $f''$ and the zeros of $f''$
   (i.e., points that are possible inflection points)
6. Determine the intervals where $f$ is concave up or concave down using the sign of $f''$
7. Determine the horizontal asymptotes

*In the future, we'll revisit this later.*
Remark: the vertical asymptotes occur at points \( a \notin \text{dom}(f) \) such that \( \lim_{x \to a} f(x) = \pm \infty \), hence we find these when we determine the domain.

**Example:**

1. \( \text{dom}(f) = \{ x \in \mathbb{R} \mid x \neq d, p \} \)
   \( \text{Z}(f) = \{ x \in \mathbb{R} \mid f(x) = 0 \} = \{ e, l, n, q \} \)

2. \( \text{dom}(f') = \{ x \in \mathbb{R} \mid x \neq d, p \} \)
   \( \text{Z}(f') = \{ b, c, j, m \} \)

3. \( f \) is increasing on \( (-\infty, b), (c, d), (d, j), (m, p) \)
   \( f \) is decreasing on \( (b, c), (j, m), (p, \infty) \)

4. \( f \) has local max at \( x = b, j, p \)
   \( f \) has local min at \( x = c, m \)

\[ \lim_{x \to d^-} f(x) = +\infty \]
\[ \lim_{x \to d^+} f(x) = -\infty \]

\[ y = f(x) \]
\[ y = L \]
(5) \( \text{dom}(f'') = \{ x \in \mathbb{R} \mid x \neq a, \beta \} \)
\( Z(f'') = \{ a, k \} \)

(6) \( f \) is concave up on \( (-\infty, a) \), \( (c, d) \), \( (k, \infty) \)
\( f \) is concave down on \( (a, c) \), \( (d, k) \)

hence \( f \) has points of inflection at \( x = a, d, k \), but not at \( x = \beta \).

(7) \( \lim_{x \to \infty} f(x) = L \)
\( \lim_{x \to -\infty} f(x) = M \)

We'll now briefly return to limits to learn \( \text{L'Hôpital's rule} \)
for evaluating such limits.
**Theorem (L'Hôpital's rule)**

If $f$ and $g$ are differentiable on an open interval containing $a$ such that

\[ f(a) = 0 = g(a) \]

Suppose further that $g'(x) \neq 0$

Then:

\[ \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \]

**Infinite version**

The theorem is true for $a = \pm \infty$ provided that, instead,

\[ \lim_{x \to \infty} f(x) = 0 = \lim_{x \to \infty} g(x) \]

or

\[ \lim_{x \to \infty} f(x) = \pm \infty \text{ and } \lim_{x \to \infty} g(x) = \pm \infty \]

and

\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \]
Examples (§ 4.5)

(31) \[ \lim_{x \to 0} \frac{\cos(x + \frac{\pi}{2})}{\sin(x)} = ? \]

Clearly \( \lim_{x \to 0} \cos(x + \frac{\pi}{2}) = 0 = \lim_{x \to 0} \sin(x) \),

so by L'Hôpital's rule:

\[ \lim_{x \to 0} \frac{\cos(x + \frac{\pi}{2})}{\sin(x)} = \lim_{x \to 0} \frac{-\sin(x + \frac{\pi}{2})}{\cos(x)} = \frac{-\sin(0)}{\cos(0)} = -1 \]

(50) \[ \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^x = ? \]

We'll use properties of logs to make use of L'Hôpital's rule.

Let \( \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^x = L \)

and take \( \ln \) of both sides; since \( \ln \) is continuous it can pass inside the limit (why?) and we get:
\[
\lim_{{x \to \infty}} \ln \left( \frac{x}{{x+1}} \right)^x = \ln(L)
\]

\[
\Rightarrow \lim_{{x \to \infty}} x \ln \left( \frac{x}{{x+1}} \right) = \ln(L)
\]

\[
\Rightarrow \lim_{{x \to \infty}} \frac{\ln \left( \frac{x}{{x+1}} \right)}{\frac{1}{x}} = \ln(L)
\]

So now the left side approaches \( \frac{0}{0} \), hence apply L'Hopital:

\[
\Rightarrow \lim_{{x \to \infty}} \left( \frac{\frac{x+1}{x} \left( \frac{x+1-x}{(x+1)^2} \right)}{-\frac{1}{x^2}} \right) = \ln(L)
\]

\[
\Rightarrow \lim_{{x \to \infty}} \left( \frac{\frac{1}{x(x+1)}}{-\frac{1}{x^2}} \right) = \ln(L)
\]

\[
\Rightarrow \lim_{{x \to \infty}} \frac{-x^2}{x(x+1)} = \ln(L)
\]

\[
\Rightarrow \lim_{{x \to \infty}} \frac{-x}{x+1} = \ln(L)
\]
Now apply L'Hopital again since the limit approaches $\infty$.

$$\lim_{x \to a} \frac{-1}{1} = \ln(L)$$

$$\Rightarrow \ln(L) = -1$$

$$\Rightarrow L = e^{-1}$$

55. \[ \lim_{x \to \frac{\pi}{2}} \frac{\cos mx}{\cos nx} \] where $m, n \in \mathbb{Z}$, $m, n \neq 0$

Since the limit approaches $\frac{0}{0}$ we may apply L'Hopital's rule:

$$\lim_{x \to \frac{\pi}{2}} \frac{-m \sin(mx)}{-n \sin(nx)}$$

Now we are finished so long as 2 does not divide $n$, i.e. $n$ is odd.
That is, if \( n \) is odd:

\[
\lim_{x \to \frac{\pi}{2}} \frac{-m \sin(mx)}{-n \sin(nx)} = \frac{-m \sin\left(\frac{mn\pi}{2}\right)}{-n \sin\left(\frac{mn\pi}{2}\right)}
\]

On the other hand, if \( n \) is even, the \( \text{L'Hôpital's rule} \) applies:

the numerator \(-n \sin(nx)\) approaches both numerator and denominator will approach 0; so apply \( \text{L'Hôpital's rule} \) again:

\[
\lim_{x \to \frac{\pi}{2}} \frac{-m^2 \cos(mx)}{-n^2 \cos(nx)} = \frac{m^2 \cos\left(\frac{mn\pi}{2}\right)}{n^2 \cos\left(\frac{mn\pi}{2}\right)}
\]

Since \( n \) is even, \( \frac{mn\pi}{2} \) is an integer multiple of \( \pi \), hence cosine is non-zero.
Now, for practice:

Try finding the critical points, etc. and sketching a graph of the preceding example.

\[
\text{line } \frac{x}{\pi} = \frac{\cos nx}{x} \quad \text{for } n \in \mathbb{Z}
\]

\[
\frac{\sin mx}{m} \quad \text{for } n \text{ odd}
\]

\[
\frac{m^2 \cos \left( \frac{m^2 \pi}{2} \right)}{m^2 \cos \left( \frac{m^2 \pi}{2} \right)}
\]
EXAMPLES OF GRAPHED FUNCTIONS (Ex 54.6)

18. \[ y = \frac{1}{3}x^3 + x^2 + 3x \]

- \( \text{dom}(y) = \mathbb{R} \)
- \( y = 0 \iff \frac{1}{3}x^3 + x^2 + 3x = 0 \), so \( x = 0 \)
  or \( \frac{1}{3}x^2 + x + 3 = 0 \),
  i.e. \( x^2 + 3x + 9 = 0 \),
  which has no real roots (check!)
  hence \( \mathbb{Z}(y) = \{0\} \)

- \( y' = x^2 + 2x + 3 \): \( \text{dom}(y') = \mathbb{R} \)
  and \( y' = 0 \iff x^2 + 2x + 3 = 0 \)
  \( \iff x = -2 \pm \sqrt{4 - 4(1)(3)} \)
  \( \iff x = -2 \pm \sqrt{-8} \)
  hence \( y'(x) \) has no real
  \( \mathbb{Z}(y') = \{\} \)
\[ y'' = 2x + 2 \quad \text{dom}(y'') = \mathbb{R} \]
and \( y'' = 0 \iff 2x + 2 = 0 \iff x = -1 \)

so \( \{y''\} = \{-1, 3\} \)

so now we can check the signs.

\( \rightarrow \) there are no critical points, so we need only check one sample for \( y' \):

\[ y'(0) = 0 + 2(0) + 3 = 3 > 0 \]

hence \( y \) is increasing everywhere.

\( \rightarrow \) there is one possible inflection:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y''(x) )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-2</td>
<td>-</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>-</td>
</tr>
</tbody>
</table>

hence \( -1 \) reflects from concave \( \downarrow \) to concave \( \uparrow \)
Finally,

\[ \lim_{x \to \infty} y(x) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} y(x) = -\infty, \]

hence there are no asymptotes.

Now we collect this into one graph:

Inflection point at \((-1, -\frac{4}{3})\)
$y = \sin(x) + \frac{1}{2} x \quad \text{over } [0, 2\pi]$ 

- $\text{dom}(y) = [0, 2\pi]$ (defined everywhere on region)

- $\mathbb{Z}(y) : \sin(x) = -\frac{1}{2} x$; clearly true for $x = 0$.

Now suppose $x \neq 0$;

Then $\frac{\sin(x)}{x} = -\frac{1}{2}$; hence this can only occur in $[\pi, 2\pi]$.

There $-1 \leq \sin(x) \leq 0$

$\Rightarrow -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq 0$

but $-\frac{1}{2} \leq -\frac{1}{x}$ for all $x \in [\pi, 2\pi]$, hence there is no such $x$.

hence $\mathbb{Z}(y) = \{0\}$
\[ y' = \cos(x) + \frac{1}{2} \]

Clearly \(\text{dom}(y') = [0, 2\pi]\)

Now if \(y' = 0\) then \(\cos(x) = -\frac{1}{2}\), which occurs only at \(x = \frac{4\pi}{3}\) and \(x = \frac{2\pi}{3}\) in \([0, 2\pi]\).

So \(\mathcal{Z}(y') = \left\{ \frac{2\pi}{3}, \frac{4\pi}{3} \right\}\)

\[ y'' = -\sin(x) \]

Clearly again \(\text{dom}(y'') = [0, 2\pi]\)

and \(\mathcal{Z}(y'') = \{0, \pi, 2\pi\}\)
Now we'll look at the sign charts

**Extrema**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$1 + \frac{1}{2} &gt; 0$</td>
<td>+</td>
</tr>
<tr>
<td>$\frac{2\pi}{3}$</td>
<td>0</td>
<td>CP</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$-1 + \frac{1}{2} &lt; 0$</td>
<td>-</td>
</tr>
<tr>
<td>$\frac{4\pi}{3}$</td>
<td>0</td>
<td>CP</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>$1 + \frac{1}{2} &gt; 0$</td>
<td>+</td>
</tr>
</tbody>
</table>

**Inflection**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f''(x)$</th>
<th>Sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>$\frac{\sqrt{2}}{2} &gt; 0$</td>
<td>+</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>0</td>
<td>PI</td>
</tr>
<tr>
<td>$\frac{3\pi}{2}$</td>
<td>$-\frac{\sqrt{2}}{2} &lt; 0$</td>
<td>-</td>
</tr>
</tbody>
</table>

*Remark: note we don't check the endpoints since inflection can only occur across a point which is contained in an open interval also contained in the domain.*
Now we collect the information in a graph of $y = \sin(x) + \frac{1}{2}x$.
\[ y = \frac{x-2}{x-3} \]

- \( \text{dom}(y) = \{ x \in \mathbb{R} \mid x \neq 3 \} \)
- \( \mathcal{Z}(y) = \{ 2 \} \)

\[ y'(x) = \frac{(1)(x-3) - (x-2)(1)}{(x-3)^2} = \frac{x-3-x+2}{(x-3)^2} = \frac{-1}{(x-3)^2} \]

- \( \text{dom}(y') = \{ x \in \mathbb{R} \mid x \neq 3 \} \)
- \( \mathcal{Z}(y') = \{ \} \)

\[ y''(x) = 2(x-3)^{-3} = \frac{2}{(x-3)^3} \]

- \( \text{dom}(y'') = \{ x \in \mathbb{R} \mid x \neq 3 \} \)
- \( \mathcal{Z}(y'') = \{ \} \)

\[ \Rightarrow \text{one possible inflection at } x = 3 \]
Now consider the sign charts

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y'(x)$</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-\frac{1}{9}$</td>
<td>-</td>
</tr>
</tbody>
</table>

so $y$ is decreasing everywhere.

And:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y''(x)$</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{2}{x^2} &lt; 0$</td>
<td>(−)</td>
</tr>
<tr>
<td>3</td>
<td>undefined</td>
<td>(±)</td>
</tr>
<tr>
<td>4</td>
<td>$2 &gt; 0$</td>
<td>(+)</td>
</tr>
</tbody>
</table>

hence there is an inflexion across $x = 3$.

Finally, we consider the asymptotes:

\[
\lim_{x \to 3^+} y(x) = +\infty \quad \lim_{x \to +\infty} y(x) = 1 \\
\lim_{x \to 3^-} y(x) = -\infty \quad \lim_{x \to -\infty} y(x) = 1
\]
Now we collect this info into our graph:

\[ y = \frac{x - 2}{x - 3} \]
(62) \[ y = \frac{x}{x^2 - 9} \]

Clearly \( y \) is defined only when \( x^2 - 9 \neq 0 \) which happens when \( x^2 - 9 = 0 \)

\[ (x-3)(x+3) \]

hence  \( \text{dom}(y) = \{ x \in \mathbb{R} \mid x \neq 3, -3 \} \)

and \( \mathcal{Z}(y) = \{ 0 \} \) is clear.

\[ y' = \frac{(1)(x^2 - 9) - (x)(2x)}{(x^2 - 9)^2} = \frac{x^2 - 9 - 2x^2}{(x^2 - 9)^2} = \frac{-x^2 + 9}{(x^2 - 9)^2} \]

Now \( \text{dom}(y') = \{ x \in \mathbb{R} \mid x \neq 3, -3 \} \)

and \( \mathcal{Z}(y') = \{ 0 \} \) since \( x^2 + 9 \neq 0 \) for all real \( x \).
\[ y'' = \frac{-2x(x^2-9)^2 + 4x(x^2+9)(x^2-9)}{(x^2-9)^4} \]

\[ = \frac{-2x(x^2-9)^2 + 4x(x^2+9)(x^2-9)}{(x^2-9)^4} \]

\[ = \frac{2x((-x^2-9) + (x^2+9))}{(x^2-9)^3} \]

\[ = \frac{2x(18)}{(x^2-9)^3} = \frac{36x}{(x^2-9)^3} \]

So then

\[ \text{dom}(y'') = \{ x \in \mathbb{R} \mid x \neq 3, -3 \} \]

and

\[ \mathbb{Z}(y'') = \{ 0, 3 \} \]
Now check the signs:

\[
\begin{array}{c|c|c}
  x & y'(x) & \text{sign} \\
  \hline
  0 & -\frac{9}{92} & \Theta \\
\end{array}
\]

So \( y \) decreases everywhere.

\[
\begin{array}{c|c|c}
  x & y''(x) & \text{sign} \\
  \hline
  -4 & -\frac{4(26)}{(16-9)^3} < 0 & \Theta \\
  -3 & 0 & \rho_+ \\
  -1 & \frac{36}{(26)^3} > 0 & \Theta \\
  0 & 0 & \rho_+ \\
  1 & \frac{36}{(28)^3} < 0 & \Theta \\
  3 & 0 & \rho_+ \\
  4 & \frac{36(4)}{(16-9)^3} > 0 & \Theta \\
\end{array}
\]

Hence \( y \) has inflection across \( x = -3, 0, \text{ and } 3 \).
Lastly we check the asymptotes:

\[ \lim_{x \to 3^-} y(x) = -\infty \quad \text{vertical} \]
\[ \lim_{x \to 3^+} y(x) = +\infty \]

\[ \lim_{x \to \pm\infty} y(x) = 0 \quad \text{horizontal} \]

Now we can graph!
\[ y = \frac{x}{x^2 + 1} \]

- Clearly \( x^2 + 1 \geq 1 \), so \( y \) is defined everywhere \( \Rightarrow \text{dom}(y) = \mathbb{R} \)
  and \( \text{dom}(y) = \{0\} \)

\[ y'(x) = \frac{1}{(x^2 + 1)^{3/2}} \]

\( \Rightarrow \text{dom}(y') = \mathbb{R} \)
  \( \Rightarrow \text{dom}(y') = \{7\} \)

\[ y''(x) = \frac{-3x}{(x^2 + 1)^{5/2}} \]

\( \Rightarrow \text{dom}(y'') = \mathbb{R} \)
  \( \Rightarrow \text{dom}(y'') = \{30\} \)

Now we check the signs:

<table>
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<th>( y'(x) )</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

so \( y \) increases everywhere.
\begin{align*}
\begin{array}{|c|c|c|}
\hline
x & y''(x) & \text{Sign} \\
\hline
-1 & \frac{3}{2^{3/2}} > 0 & \bigoplus \\
0 & 0 & (\text{PT}) \\
1 & \frac{-3}{2^{3/2}} < 0 & \bigotimes \\
\hline
\end{array}
\end{align*}

hence an inflection from concave up to concave down at \( x = 0 \).

Finally, the asymptotes:

\begin{align*}
\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} & \rightarrow \sqrt{\lim_{x \to \infty} \frac{x^2}{x^2 + 1}} = \sqrt{1} = 1 \\
\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 1}} & \rightarrow -\sqrt{\lim_{x \to -\infty} \frac{x^2}{x^2 + 1}} = -\sqrt{1} = -1
\end{align*}
and the graph: