Q: How far does a particle with constant velocity \( v(t) = a \) m/sec. go in \( b \) sec.?

A: \((a \text{ m/sec}) \cdot (b \text{ sec}) = ab \text{ meters} \)

See the graph of \( v(t) = a \):

Note that the area under the graph \( v(t) = y \) for \( 0 \leq t \leq b \) is precisely \( ab \)
the distance travelled by the particle.

This suggests the same answer to a more general question:

Q: How far does a particle with velocity \( v(t) \) go in \( b \) sec?

and

in fact, the answer is the same:

A: the area under the graph

\[ y = v(t) \]

\[ \int_a^b v(t) \, dt \]

\[ 0 \leq t \leq b \]

Hence: we want to understand the area under a graph!

which we denote by

\[ \int_a^b f(x) \, dx \]
As in the case of the derivative, we can define this quantity as a limit.

We will approach this by considering approximations of the area via Riemann sums.

To define such things we need to learn

**Summation notation**:\[ \sum_{j=m}^{n} a_j = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \]

\[ \sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 \]
\[ \sum_{k=2}^{5} f(k) = f(2) + f(3) + f(4) + f(5) \]
Important properties:

- **Linearity:**
  \[ \sum_{i=n}^{m} (a_i + b_i) = \sum_{i=n}^{m} a_i + \sum_{i=n}^{m} b_i \]
  \[ \sum_{i=n}^{m} ka_i = k \sum_{i=n}^{m} a_i \]

- **Sum of constants:**
  \[ \sum_{i=n}^{m} k = (m-n+1)k \]
  (adding up \( k \) \((m-n+1) \) times)

- **Power sums:**
  \[ \sum_{j=1}^{N} j = \frac{N(N+1)}{2} \]
  \[ \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6} \]
  \[ \sum_{j=1}^{N} j^3 = \frac{N^2(N+1)^2}{4} \]
Some examples:

33. \[ \sum_{j=101}^{200} j = ? \]

See that \[ \sum_{j=1}^{200} j = \sum_{j=1}^{100} j + \sum_{j=101}^{200} j \]

\[ \Rightarrow \frac{200(201)}{2} = \frac{100(101)}{2} + \sum_{j=101}^{200} j \]

\[ \Rightarrow 100(201) - 50(101) = \sum_{j=101}^{200} j \]

[Apply partial sum formulae]

36. \[ \sum_{k=1}^{30} 4k - 3 = ? \]

See that \[ \sum_{k=1}^{30} 4k - 3 = 4 \sum_{k=1}^{30} k - \sum_{k=1}^{30} 3 \]

\[ = 4 \frac{(30)(31)}{2} - 30(3) \]

\[ = 30(2(31) - 3) = 30(62 - 3) \]

\[ = \boxed{(59)(30)} \]
\[ \sum_{k=1}^{200} k^3 = \sum_{k=1}^{100} k^3 + \sum_{k=101}^{200} k^3 \]

\[ \frac{(200)^2(101)^2}{4} - \frac{(100)^2(101)^2}{4} = \sum_{k=101}^{200} k^3 \]

Suppose \( f \) is a function with the following values at \( a_1, a_2, a_3, a_4 \).

What is the area of the corresponding rectangular regions?

\[ \text{Area} = \left[ f(a_1) a_1, f(a_2) (a_2 - a_1), f(a_3) (a_3 - a_2), f(a_4) (a_4 - a_3) \right] \]

\[ \text{Total area} = \sum_{i=1}^{4} (a_i - a_{i-1}) f(a_i) \]

Where we take \( a_6 = 0 \).

\[ \text{an example of a Riemann sum} \]
Riemann Sums

The Riemann sums provide estimations of the area $\int_a^b f(x) \, dx$.

Consider an interval $[a, b]$ and a partition

$$x_j = a + \frac{j(b-a)}{N}$$

for $j = 0, \ldots, N$.
Then we can define the corresponding Riemann sums for the area under $y = f(x)$ over the interval $[a, b]$:

**Right:**

$$R_N = \frac{(b-a)}{N} \sum_{j=1}^{N} f(x_j)$$

**Left:**

$$L_N = \frac{(b-a)}{N} \sum_{j=0}^{N-1} f(x_j)$$

**Midpoint:**

$$M_N = \frac{(b-a)}{N} \sum_{j=0}^{N-1} f\left(\frac{x_j + x_{j+1}}{2}\right)$$
Theorem (Riemann)

If $f$ is continuous on $[a, b]$; then:

$$
\lim_{N \to \infty} R_N = \lim_{N \to \infty} L_N = \lim_{N \to \infty} M_N = \int_a^b f(x) \, dx
$$

Examples:

14. $f(x) = \sqrt{x}$ over $[0, 4]$

**Partition:** $x_j = \frac{j}{N} 4$

- $R_4 = \frac{4}{4} \sum_{j=1}^{4} f\left(\frac{j}{4}\right) = \frac{4}{4} \sum_{j=1}^{4} f(j) = 1 + \sqrt{2} + \sqrt{3} + 2$

- $L_4 = \frac{4}{4} \sum_{j=0}^{3} f\left(\frac{j}{4}\right) = \frac{4}{4} \sum_{j=0}^{3} f(j) = 0 + 1 + \sqrt{2} + \sqrt{3}$

- $M_4 = \frac{4}{4} \sum_{j=0}^{3} f\left(\frac{1}{2}\left(\frac{j}{4} + (j+1)\frac{1}{4}\right)\right) = \sum_{j=0}^{3} f\left(\frac{2j+1}{2}\right) = -\frac{1}{2} + \sqrt{3} + \sqrt{5} + \sqrt{7}$
Compare these estimators!

Clearly here,

\[ L_4 < \int_0^4 f(x) \, dx < R_4 \]

and,

\[ L_4 < M_4 < R_4 \]

(try seeing this for yourself)

**FACT:**

If \( f(x) \) is increasing then \( L_n \leq M_n \leq R_n \)
(16) \( f(x) = x^2 \), \([0, 2]\)

**Partition:** \( x_j = j \cdot \frac{2}{N} \)

**Riemann Sums:**
\[
R_N = \frac{2}{N} \sum_{j=1}^{N} f(x_j) = \frac{2}{N} \sum_{j=1}^{N} \frac{4j^2}{N^2} = \frac{8}{N^3} \sum_{j=1}^{N} j^2
\]
\[
= \frac{8}{N^3} \left( \frac{N(N+1)(2N+1)}{6} \right)
\]

\[
L_N = \frac{2}{N} \sum_{j=0}^{N-1} f(x_j) = \frac{2}{N} \sum_{j=0}^{N-1} \frac{4j^2}{N^2} = \frac{8}{N^3} \sum_{j=1}^{N-1} j^2
\]

Notice index!
\[
= \frac{8}{N^3} \left( \frac{(N-1)N(2N-1)+1}{6} \right)
\]

Now see that, applying L'Hôpital's:

\[
\lim_{N \to \infty} L_N = \lim_{N \to \infty} R_N = \frac{8}{3}
\]

\[
= \int_0^2 x^2 \, dx
\]
(e.g. 3) \( f(x) = 3x^2 - 2x \); find \( \int_1^2 f(x) \, dx \)

as a limit of right Riemann sums

See that length of \([1, 2]\) is 1

and partition is \( x_j = 1 + \frac{j}{N} \)

Then:

\[
R_N = \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \frac{1}{N} \sum_{j=1}^{N} \left[ 3\left(1 + \frac{j}{N}\right)^2 - 2\left(1 + \frac{j}{N}\right) \right]
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \left[ 3\left(1 + \frac{j^2}{N^2} + \frac{2j}{N}\right) - 2 - \frac{2j}{N} \right]
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \left[ 3 + \frac{6j}{N} + \frac{3j^2}{N^2} - 2 - \frac{2j}{N} \right]
\]

\[
= \frac{1}{N} \sum_{j=1}^{N} \left[ 1 + \frac{4j}{N} + \frac{3j^2}{N^2} \right]
\]

\[
= \frac{1}{N} \left[ \sum_{j=1}^{N} 1 + \frac{4}{N} \frac{N}{2} \frac{j}{j} + \frac{3}{N^2} \frac{N}{2} \sum_{j=1}^{N} j^2 \right]
\]

\[
= \frac{1}{N} \left[ N + \frac{4(N)(N+1)}{N^2} + \frac{3(N)(N+1)(2N+1)}{6N^2} \right]
\]
\[
\left[ 1 + \frac{4(N+1)}{2N} + \frac{3(N+1)(2N+1)}{N^2} \right]
\]

Then:

\[
\lim_{N \to \infty} R_N = \lim_{N \to \infty} \left[ 1 + \frac{4N+1}{2N} + \frac{6N^2+9N+3}{N^2} \right]
\]

\[
= 1 + \frac{4}{2} + \frac{6}{1} = 1 + 2 + 6 = 9
\]

Here \( \int_1^2 f(x) \, dx = 9 \)
Now, since we can define \( \int_a^b f(x) \, dx \) as a limit of sums \( \sum_{i=1}^{N} \), we get many similar properties.

1. **Constant**: \( \int_a^b M \, dx = M(b-a) \)

2. **Linearity**:
   \[
   \int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \\
   \int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx
   \]

3. **Interval splitting**
   \[
   \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
   \]

4. **Signed intervals**
   \[
   \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx
   \]

(see HW #9 sol'n examples)
\[
\text{note: } \int_a^b f(x) \, dx = \text{signed area}
\]

for functions that aren't \( f(x) \geq 0 \)

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

\[-(\text{Area } 2)\] (Area 2)