Math 20B
Midterm Exam 1.
April 24, 2019

Exam rules:

1. No electronic devices of any kind are allowed during this exam.

2. You may use one 2-sided US Letter sized page of notes, but no books or other assistance are allowed during this exam.

3. If you violate these instructions or communicate in any way with any other student during this exam, you will receive a zero on the exam, and the zero will not be dropped when calculating your cumulative course average.

Instructions:

1. Write your first and last name at the top of every page of this exam.

2. Write your solutions clearly and legibly. No credit will be given for illegible solutions.

3. Show the work leading to your solutions. Partial credit can only be given based on the work you show.

4. Please put a box around your final answer to each part.

→ If any question is not clear, ask for clarification.
1. (10 points)

Consider the following two curves:

\[ y = \frac{19 - 5x}{4} \quad \text{and} \quad y = x^2 - 4x + 4 = (x-2)^2 \]

(a) Find the area of the region between the two curves.

\[ \left[ \frac{-5x + 19}{4} - (x-2)^2 \right] \]

- **Intercept point:**

  \[ \frac{19 - 5x}{4} = x^2 - 4x + 4 \]

  \[ 19 - 5x = 4x^2 - 16x + 16 \Rightarrow 0 = 4x^2 - 11x - 3 \]

  \[ x = \frac{11 \pm \sqrt{121 - 4(4)(-3)}}{2(4)} = \frac{11 \pm \sqrt{121 + 48}}{8} = \frac{11 \pm \sqrt{169}}{8} = \frac{11+13}{8}, \frac{11-13}{8} \]

- **Area:**

  \[ \text{area} = \int_{-1}^{3} \left[ \frac{-5x + 19}{4} - (x-2)^2 \right] \, dx = 3, \frac{1}{4} \]

  \[ = \int_{-\frac{1}{4}}^{3} \left[ -\frac{5x}{4} + \frac{19}{4} - x^2 + 4x - 4 \right] \, dx = \int_{-\frac{1}{4}}^{3} \left[ -\frac{x^2}{8} + \frac{11x}{4} + \frac{3}{4} \right] \, dx \]

  \[ = \left[ -\frac{x^3}{3} + \frac{11x^2}{8} + \frac{3x}{4} \right]_{-\frac{1}{4}}^{3} \]

  \[ = \left[ -3^2 + \frac{27}{8} + \frac{9}{4} \right] - \left[ -\frac{1}{3} \left( \frac{8}{36} \right)^3 + \frac{11}{8} \left( \frac{27}{8} \right) + \frac{3}{4} \left( \frac{3}{4} \right) \right] \]

  \[ = -3^2 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{3^3} + \frac{11}{8} \cdot \frac{27}{8} + \frac{3}{4} \cdot \frac{3}{4} \right] \]

  \[ = -9 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{27} + \frac{11}{8} \cdot \frac{27}{8} + \frac{3}{4} \cdot \frac{3}{4} \right] \]

  \[ = -9 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{27} + \frac{27}{8} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \right] \]

  \[ = -9 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{27} + \frac{27}{8} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \right] \]

  \[ = -9 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{27} + \frac{27}{8} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \right] \]

  \[ = -9 + \frac{27}{8} + \frac{9}{4} - \left[ -\frac{1}{3} \cdot \frac{8}{27} + \frac{27}{8} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{3}{4} \right] \]
(b) Find the volume of the solid whose base is the region between the two curves and whose cross-sections parallel to the y-axis are triangular with equal length base and height.

\[ A(x) = \frac{1}{2} b^2 \quad \left( = \frac{1}{2} bh \text{ with } b = h \right) \]

So, the volume is

\[
\int_{-\frac{1}{4}}^{3} A(x) \, dx = \int_{-\frac{1}{4}}^{3} \left[ -\frac{5x}{4} + \frac{19}{4} - (x-2)^2 \right] \, dx
\]
2. (10 points) Setup, but do not evaluate, an integral giving the volume of the solid obtained from revolving the region in question 1 about the $x$-axis.

\[ \pi \int_{-1/4}^{3} \left[ \left( \frac{19-5x}{4} \right)^2 - (x-2)^2 \right] \, dx \]
3. (10 points) Find the general antiderivative of the following function using integration by parts:

\[ f(x) = 5x \ln(x^2) \]

Consider \( \int f(x) \, dx \):

Let \( u = \ln(x^2) \) and \( dv = 5x \, dx \)

Then \( du = \frac{1}{x^2} (2x) \, dx \) and \( v = \frac{5x^2}{2} \)

So we have

\[
\int f(x) \, dx = \int u \, dv = uv - \int v \, du
\]

\[
= \ln(x^2) \left( \frac{5x^2}{2} \right) - \int \frac{5x^2}{2} \left( \frac{2x}{x^2} \right) \, dx
\]

\[
= \frac{5\ln(x^2)x^2}{2} - \frac{5x^2}{2} \int x \, dx
\]

\[
= \frac{5\ln(x^2)x^2}{2} - \frac{5x^2}{2} + C
\]
4. (10 points) Find the area inside the polar curve \( r = 7 \cos(4\theta) \) and outside a circle of radius 3 centered at the origin.

(You may find the identity \( \cos^2(x) = \frac{1 + \cos(2x)}{2} \) useful.)

\[
\text{Area} = \int_{0}^{2\pi} \left( (7 \cos(4\theta))^2 - (3)^2 \right) \, d\theta
= \int_{0}^{2\pi} [49 \cos^2(4\theta) - 9] \, d\theta
= 49 \int_{0}^{2\pi} \cos^2(4\theta) \, d\theta - 9 \int_{0}^{2\pi} \, d\theta
= 49 \int_{0}^{2\pi} \left( \frac{1 + \cos(8\theta)}{2} \right) \, d\theta - 9 [2\pi]
= \frac{49}{2} \int_{0}^{2\pi} (1 + \cos(8\theta)) \, d\theta - 18\pi
= \frac{49}{2} \left[ \frac{\theta + \sin(8\theta)}{8} \right]_{0}^{2\pi} - 18\pi
= \frac{49}{2} \left[ 2\pi + \frac{\sin(16\pi)}{8} - 0 - \frac{\sin(0)}{8} \right] - 18\pi = 9\pi - 18\pi = 31\pi
\]