1 Complex numbers

The real numbers \( \mathbb{R} \) are the set where we do calculus because of a nice property called completeness: (a kind of function)

every sequence of rational numbers (fraction of integers) has a limit

The set of rational numbers is often denoted \( \mathbb{Q} \), and set of integers denoted \( \mathbb{Z} \)

The completeness property can also be stated that:

\( \mathbb{R} \) is the limit closure of \( \mathbb{Q} \)
Now, there is a different type of closure over \( \mathbb{Q} \) called algebraic closure:

A set is algebraically closed over \( \mathbb{Q} \) if it contains all the roots of polynomials with coefficients in \( \mathbb{Q} \); i.e.

If \( p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \)

with \( a_0, \ldots, a_n \) in \( \mathbb{Q} \)

then the set contains all the numbers \( r_1, \ldots, r_n \) with \( p(r_i) = 0 \)

Big idea: \( \mathbb{R} \) is NOT algebraically closed over \( \mathbb{Q} \)!

See \( p(x) = x^2 + 1 \) has roots \( \pm i \)

Not in \( \mathbb{R} \).
But, we can define a new set called:

\[ C = \mathbb{R}(i) = \{a + bi \mid a, b \in \mathbb{R}\} \]

the complex numbers

\[ i := \sqrt{-1} \]

4. These are algebraically closed over \( \mathbb{Q} \).

**Technical Remark:** \( C \) is actually the algebraic closure of \( \mathbb{R} \), not \( \mathbb{Q} \).

The algebraic closure of \( \mathbb{Q} \) is actually smaller than \( C \)!

In other words, over \( C \) we can solve all polynomials!

(even more with \( \hat{a} = \sqrt{-1} \))

\[ \text{Re} \{a + ib\} \quad \text{Real part of \( \sqrt{-1} \)} \]

\[ \text{Im} \{a + ib\} \quad \text{Imaginary part} \]
Numbers in $\mathbb{C}$ can be...

- added: $(a+ib) + (c+id) = (a+c) + (b+d)i$
- scaled by reals: $c(a+ib) = ca + icb$
- multiplied: $(a+ib)(c+id) = ac + i(ad + bc) + i^2bd = ac - bd + i(ad + bc)$

$\mathbb{C}$ can also be thought of as a plane! (Note $\mathbb{C}$ carries the data of $\mathbb{R}^2$...)

In this way they carry a length:

$$|z| = \sqrt{a^2 + b^2}$$
related to the length is the conjugate:

$$z = a + ib \text{ has conjugate } \overline{z} := a - ib$$

Ipsenent fact: $$\overline{z} \cdot z = |z|^2$$

This can be used to find the inverse of complex number $$\frac{1}{z}$$

$$\overline{z}' = (a + ib)^{-1} = \frac{1}{a + ib} \cdot \frac{a - ib}{a + ib}$$

$$= \frac{a - ib}{a^2 + b^2} = \frac{a}{|z|^2} - \frac{ib}{|z|^2}$$
2. Complex exponentials

By Euler, we define

\[ e^{i\theta} = \cos \theta + i \sin \theta \]

In particular, every complex number has a polar form:

\[ z = x + iy = r \cos(\theta) + ir \sin(\theta) = r(e^{i\theta}) \]

where

\[ r = |z| \]

Multiply \( z \cdot w = \begin{cases} \text{scale by } r & \text{rotate } w \text{ by } \theta \end{cases} \]
From here we can also get:

\[
\begin{align*}
\cos(-\theta) &= \cos(\theta) + i\sin(\theta) \\
\sin(-\theta) &= \cos(\theta) - i\sin(\theta)
\end{align*}
\]

- Cosine is even
- Sine is odd

and combining with Euler we have complex exponential versions for sine and cosine:

\[
\begin{align*}
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{align*}
\]
Now: we obtain the "double angle" identities

\[ \cos^2(\theta) = \frac{1}{4} \left( e^{i\theta} e^{-i\theta} + e^{i\theta} e^{-i\theta} + e^{2i\theta} + e^{-2i\theta} \right) \]

\[ = \frac{1}{4} \left( 1 + 1 + e^{2i\theta} + e^{-2i\theta} \right) \]

\[ = \frac{2}{4} + \frac{e^{2i\theta} + e^{-2i\theta}}{4} \]

\[ = \frac{1}{2} + \frac{1}{2} \left( \frac{e^{2i\theta} + e^{-2i\theta}}{2} \right) \]

\[ \Rightarrow \quad \cos^2(\theta) = \frac{1}{2} \left( 1 + \cos(2\theta) \right) \]

\[ \sin^2(\theta) = \frac{1}{4} \left( e^{i\theta} e^{-i\theta} - e^{i\theta} e^{-i\theta} + e^{2i\theta} + e^{-2i\theta} \right) \]

\[ = \frac{1}{4} \left( -1 - 1 + e^{2i\theta} + e^{-2i\theta} \right) \]

\[ = \frac{1}{2} - \frac{1}{4} \left( e^{2i\theta} + e^{-2i\theta} \right) \]

\[ = \frac{1}{2} - \frac{1}{4} \left( \frac{e^{2i\theta} + e^{-2i\theta}}{2} \right) \]

\[ \Rightarrow \quad \frac{1}{2} \left( 1 - \cos(2\theta) \right) \]

\[ \sin^2(\theta) = \frac{1}{2} \left( 1 - \cos(2\theta) \right) \]
and from there, the circle:

3) \( \sin^2 \theta + \cos^2 \theta \)

\[
= \frac{1}{2} (1 + \cos(2\theta)) + \frac{1}{2} (1 - \cos(2\theta))
\]

\[
= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cos(2\theta) - \frac{1}{2} \cos(2\theta)
\]

\[
\Rightarrow \quad \sin^2 \theta + \cos^2 \theta = 1
\]

and alternate double angles:

4) \( \cos^2 \theta - \sin^2 \theta \)

\[
= \frac{1}{2} (1 + \cos(2\theta)) - \frac{1}{2} (1 - \cos(2\theta))
\]

\[
= \frac{1}{2} - \frac{1}{2} + \cos(2\theta) + \cos(2\theta)
\]

\[
= \frac{1}{2} \cdot \frac{\cos(2\theta) + \cos(2\theta)}{2}
\]

\[
\Rightarrow \quad (\cos^2 \theta - \sin^2 \theta) = \cos(2\theta)
\]
\[ \sin(2\theta) = \frac{e^{i2\theta} - e^{-i2\theta}}{2i} \]

\[ = \frac{(e^{i\theta} - e^{-i\theta})(e^{i\theta} + e^{-i\theta})}{2i} \]

\[ = \frac{2}{2i} (e^{i\theta} - e^{-i\theta}) (e^{i\theta} + e^{-i\theta}) \]

\[ = 2 \sin(\theta) \cos(\theta) \]

\[ \Rightarrow \sin(2\theta) = 2 \sin(\theta) \cos(\theta) \]
Finally, we can use properties of exponents to write the general identity for adding angles:

\[
e^{i(x+\beta)} = e^{ix}e^{i\beta}
\]

\[
\cos(x+\beta) + i\sin(x+\beta) = (\cos(x) + i\sin(x))(\cos(\beta) + i\sin(\beta))
\]

\[
= \cos(x)\cos(\beta) + i\sin(x)\cos(\beta) + i\sin(x)\sin(\beta) - \sin(x)\sin(\beta)
\]

\[
= \cos(x)\cos(\beta) - \sin(x)\sin(\beta) + i(\sin(x)\cos(\beta) + \sin(\beta)\cos(x))
\]

\[
\Rightarrow \cos(x+\beta) = \cos(x)\cos(\beta) - \sin(x)\sin(\beta)
\]

\[
\sin(x+\beta) = \sin(\beta)\cos(x) + \sin(x)\cos(\beta)
\]