1. Trigonometric integrals

Integrals of the form:

- $\int \sin^3(x) \cos^3(x) \, dx$
- $\int \sin^3(x) \, dx$
- $\int \tan^2(x) \sec^3(x) \, dx$
- $\int \tan^4(x) \, dx$
- $\int \sec^3(x) \, dx$
- $\int \sin^2(x) \cos^2(x) \, dx$

can be evaluated using a combination of substitution and trigonometric identities.
In all of these cases we want to use trig identities to make a given substitution fit.

Consider the examples

1. \[ \int \sin^2(x) \cos(x) \, dx = (\star) \]

   Let \( u = \sin(x) \)
   Then \( du = \cos(x) \, dx \) and we have:
   \[ (\star) = \int u^2 \, du \]
   and:
   \[ = \frac{u^3}{3} + C \quad \Rightarrow \quad \frac{\sin^3(x)}{3} + C \]

where the substitution fits without any trouble!
\[ \int \sin^2(x) \cos^3(x) \, dx = (*) \]

Again let \( u = \sin(x) \) and we get \( du = \cos(x) \, dx \).
Now we can substitute part way:

\[(*) = \int u^2 \cos^2(x) \cos(x) \, dx \]

\[= \int u^2 \cos^2(x) \, du = (***) \]

We run into difficulty since this doesn’t fit with our substitution, but we can apply the identity

\[ \sin^2(x) + \cos^2(x) = 1 \]

to convert instances of \( \cos^2(x) \) into \( 1 - \sin^2(x) \), which fits with our choice of \( u \).
So now:

\[
(\text{**}) = \int u^2 \cos^4(x) \, dx = \int u^2 (1-\sin^2(x)) \, dx \\
= \int u^2 (1-u^2) \, du \\
\overset{\text{now we can apply}}{\Rightarrow} \int (u^2-u^4) \, du \\
= \frac{u^3}{3} - \frac{u^5}{5} + C
\]

```
\text{``result''} = \left[ \frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} \right] + C
```
In fact, for all such examples, we have a nice rule of thumb for choice of substitution & identity:

$$\int \sin^m(x) \cos^n(x) \, dx$$

<table>
<thead>
<tr>
<th>Exponents</th>
<th>Sub</th>
<th>Identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>m odd</td>
<td>$u = \cos(x)$</td>
<td>$\sin^2(x) + \cos^2(x) = 1$</td>
</tr>
<tr>
<td>n odd</td>
<td>$u = \sin(x)$</td>
<td></td>
</tr>
</tbody>
</table>
| m and n even | Reduce even powers with i.d.'s: \[ \begin{align*}
\sin^2(x) &= \frac{1 - \cos(2x)}{2} \\
\cos^2(x) &= \frac{1 + \cos(2x)}{2}
\end{align*} \] | |
we can also apply the sec/tan version of the circle identity to integrals of their products:

\[ \tan^2(x) + 1 = \sec^2(x) \]

For example:

\[ \int \tan^3(x) \sec^4(x) \, dx = (*) \]

let \( u = \tan(x) \), and then \( du = \sec^2(x) \, dx \). So:

\[ (*) = \int u^3 \sec^2(x) \, du \]

now apply the identity + substitution

\[ = \int u^3(\tan^2(x) + 1) \, du = \int u^3(u^2+1) \, du \]
\[ \int (u^5 + u^3) \, du = \frac{u^6}{6} + \frac{u^4}{4} + C = \frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C \]

*Some Remarks on Sec/Tan Integrals:*

1. \[ \int \tan(x) \, dx = \ln |\sec(x)| + C \]
   (Verify w/ \( \tan(x) = \frac{\sin(x)}{\cos(x)} \) & sub.)

2. \[ \int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C \]
   (Harder to see! Think: multiply by \( \frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)} = 1 \))

3. Usual Rules:
   (a) \[ \int \sec^2(x) \, dx = \tan(x) + C \]
   (b) \[ \int \sec(x) \tan(x) \, dx = \sec(x) + C \]
**Rule of Thumb for \( \sec^n(x) \tan^m(x) \) Integrals:**

\[
\int \tan^m(x) \sec^n(x) \, dx
\]

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<td>n even, m either</td>
<td>( u = \tan(x) )</td>
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<td></td>
</tr>
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<td>( u = \sec(x) )</td>
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**Note:** \( \int \sec^n(x) \, dx, \int \tan^n(x) \, dx, \int \cos^n(x) \, dx \), and \( \int \sin^n(x) \, dx \) are all reducible to integration by parts!
The most important of these reduction formulae is for \( \secant \): (for doing 
\( \tan \) integrals)

\[
\int \sec^m(x) \, dx
\]

Let \( u = \sec^{m-2}(x) \) \implies \( du = (m-2) \sec(x) \tan(x) \sec(x) \tan(x) \) 
\( dv = \sec^2(x) \, dx \) \implies \( v = \tan(x) \).

Then \( \star \) (IBP)

\[
\int \sec^m(x) \, dx = uv - \int v \, du
\]

\[
= \sec^{m-2}(x) \tan(x) - (m-2) \int \sec^{m-2}(x) \tan^2(x) \, dx
\]

\( \text{(apply } \text{circle ID)} \)

\[
= \sec^{m-2}(x) \tan(x) - (m-2) \int (\sec^m(x) - \sec^{m-2}(x)) \, dx
\]

Now solve for \( \int \sec^m(x) \, dx \):

\[
\int \sec^m(x) \, dx = \sec^{m-2}(x) \tan(x) - (m-2) \int \sec^m(x) \, dx + (m-2) \int \sec^{m-2}(x) \, dx
\]

\[
\Rightarrow [(m-2)+1] \int \sec^m(x) \, dx = \sec^{m-2}(x) \tan(x) + (m-2) \int \sec^{m-2}(x) \, dx
\]

\[
\Rightarrow \int \sec^m(x) \, dx = \frac{\sec^{m-2}(x) \tan(x)}{m-1} + \frac{(m-2)}{m-1} \int \sec^{m-2}(x) \, dx
\]
Reduction formulae for \( \int \tan^m(x) \, dx \) and \( \int \cos^m(x) \, dx \) are derived at the end of §7.1 in the book. Derive one for \( \int \tan^m(x) \, dx \) as an exercise.

* A complete list of reduction and other trig integrals can be found at the end of §7.2 (on page 385).
(2) Trigonometric Substitution

A strategy for doing integrals containing (deg 2) polynomials under roots is to substitute the variable w/ a trig function, and thus turning the integral into a trig integral!

Consider the following basic example:

1) \[ \int x^4 \sqrt{1-x^2} \, dx \]

Let \( x = \sin(t) \)

Then \( dx = \cos(t) \, dt \)

Now substitute:

\[ \int x^4 \sqrt{1-x^2} \, dx = \int \sin^4(t) \sqrt{1-\sin^2(t)} \cos(t) \, dt \]

*now we can use trig identities to simplify this part!
\[ \int \sin^4(t) \sqrt{\cos^2(t)} \cos(t) \, dt \]

\[ = \int \sin^4(t) \cos(t) \cos(t) \, dt \]

\[ = \int \sin^4(t) \cos^2(t) \, dt \]

Now we have a trig integral, so let's solve. We must reduce the even powers using the identities:

\[ \sin^2(t) = \frac{1}{2} (1 - \cos(2t)) \]

\[ \cos^2(t) = \frac{1}{2} (1 + \cos(2t)) \]

so we get:

\[ = \int (\sin^2(t))^2 \left( \frac{1}{2} (1 + \cos(2t)) \right) \, dt \]

\[ = \int \left( \frac{1}{2} (1 - \cos(2t)) \right)^2 \left( \frac{1}{2} (1 + \cos(2t)) \right) \, dt \]

\[ = \frac{1}{2^3} \int (1 - \cos(2t))^2 (1 + \cos(2t)) \, dt \]

\[ = \frac{1}{8} \int (1 - \cos(2t))(1 - \cos^2(2t)) \, dt \]
\[
\frac{1}{8} \int (1 - \cos^2(2t) - \cos(2t) + \cos^3(2t)) \, dt
\]

\[
= \frac{1}{8} \int \, dt - \frac{1}{8} \int \cos^2(2t) \, dt - \frac{1}{8} \int \cos(2t) \, dt + \frac{1}{8} \int \cos^3(2t) \, dt
\]

\[
= \frac{1}{8} \left( \frac{1}{2} \int (1 + \cos(4t)) \, dt \right) - \frac{1}{8} \int \cos(2t) \, dt + \frac{1}{8} \int \cos^3(2t) \, dt
\]

apply identity 2nd time

\[
= \frac{1}{8} \left( \frac{1}{16} \int (1 + \cos(4t)) \, dt \right) - \frac{1}{8} \int \cos(2t) \, dt + \frac{1}{8} \int \cos^3(2t) \, dt
\]

\[
= \frac{1}{8} \left( -\frac{1}{16} \int \cos(4t) \, dt - \frac{1}{8} \int \cos(2t) \, dt + \frac{1}{8} \int \cos^3(2t) \, dt \right)
\]

\[
= \frac{1}{8} \left( -\frac{1}{16} \left[ \frac{\sin(4t)}{4} \right] - \frac{1}{8} \left[ \frac{\sin(2t)}{2} \right] + \frac{1}{8} \int \cos^3(2t) \, dt \right)
\]

\[
\int \cos^3(2t) \, dt = \int \cos(2t) \cos^2(2t) \, dt
\]

\[
= \int \cos(2t)(1 - \sin^2(2t)) \, dt
\]

(\text{Let } u = \sin(2t) \text{ and } du = 2\cos(2t) \, dt)

\[
\text{then: } \int (1 - u^2) \frac{du}{2} = \frac{1}{2} \left[ u + \frac{u^3}{3} \right]
\]

\[
= \frac{1}{2} \left[ \sin(2t) - \frac{\sin^3(2t)}{3} \right] + C
\]
Combining, we get:

\[ (x) = \frac{1}{16} - \frac{1}{16} \left( \frac{\sin(4t)}{4} \right) - \frac{1}{8} \left( \frac{\sin(2t)}{2} \right) + \frac{1}{8} \left( \frac{1}{2} \left( \sin(2t) - \frac{\sin^3(2t)}{3} \right) \right) + C \]

\[ = \frac{1}{16} - \frac{\sin(4t)}{64} - \frac{\sin(2t)}{16} + \frac{\sin(2t)}{16} - \frac{\sin^3(2t)}{48} + C \]

\[ = \frac{1}{16} - \frac{\sin(4t)}{64} - \frac{\sin^3(2t)}{48} + C \]

\[ \downarrow \]

*But wait! This is not the answer...*

Recall, the original integral was in terms of $x$ and we made the sub $x = \sin(t)$

Now we need to fix the answer so we may re-substitute for $x$. 
Now we use the identities
\[
\begin{align*}
\sin(2t) &= 2 \sin(t) \cos(t) \\
\cos(2t) &= \cos^2(t) - \sin^2(t)
\end{align*}
\]
and see:
\[
\begin{align*}
\frac{1}{16} - \frac{\sin(4t)}{64} - \frac{\sin^3(2t)}{48} + C \\
= & \quad \frac{1}{16} - \frac{2 \sin(2t) \cos(2t)}{64} - \frac{(2 \sin(t) \cos(t))^3}{48} + C \\
= & \quad \frac{1}{16} - \frac{1}{32} \left( 2 \sin(t) \cos(t) \cdot (\cos^2(t) - \sin^2(t)) \right) \\
& \quad - \frac{8 \sin^3(t) \cos^3(t)}{48} + C \\
= & \quad \frac{1}{16} - \frac{2}{32} \left( \sin(t) \cos(3t) - \sin^3(t) \cos(t) \right) \\
& \quad - \frac{\sin^3(t) \cos^3(t)}{6} + C \\
= & \quad \frac{1}{16} - \frac{\sin(t) \cos(3t)}{16} + \frac{\sin^3(t) \cos(t)}{16} - \frac{\sin^3(t) \cos^3(t)}{6} + C
\end{align*}
\]
We're still not finished because we need a way to deal with \( \cos(x) \).

See that we used \( x = \sin(t) \) as the sub; therefore, \( t \) is the angle in a triangle like so:

\[
\begin{align*}
1 & \quad \sin(t) = \frac{x}{1} = \frac{\text{opposite}}{\text{hypotenuse}} \\
\sqrt{1-x^2} & \quad \cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{1-x^2}}{1}
\end{align*}
\]

Therefore, \( \cos(t) = \frac{\sqrt{1-x^2}}{1} \).

So now we can resub:

\[
\begin{align*}
\sin(t) & = x \\
\cos(t) & = \sqrt{1-x^2}
\end{align*}
\]

and the solution is ....
\[
\begin{align*}
= & \left[ \frac{1}{16} - \frac{1}{16} x(1-x^2) \right] + \frac{1}{16} x^3(1-x^2) - \frac{x^3(1-x^2)^3}{6} + C \\
\end{align*}
\]

Recall the steps:

1. Substitute with appropriate trig function
2. Simplify integrand with trig identity
3. Do trig integral! * 
4. Use identities and triangle to adjust answer for re-substitution

\* See following chart
**Rules for trig sub:**

<table>
<thead>
<tr>
<th>Form of integrand</th>
<th>Choice of sub</th>
<th>Identify</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{a^2 - x^2}$</td>
<td>$x = a \sin(t)$</td>
<td>$\sin^2(t) + \cos^2(t) = 1$</td>
</tr>
<tr>
<td>$\sqrt{a^2 + x^2}$</td>
<td>$x = a \tan(t)$</td>
<td>$\tan^2(t) + 1 = \sec^2(t)$</td>
</tr>
<tr>
<td>$\sqrt{x^2 - a^2}$</td>
<td>$x = a \sec(t)$</td>
<td></td>
</tr>
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If the integrand isn't a binomial under a radical, then you'd need to complete the square to rewrite in correct form.
Please also remember the inverse trig antiderivative rules:

\[
\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a} \arcsin\left(\frac{x}{a}\right) + C
\]

\[
\int \frac{-dx}{\sqrt{a^2-x^2}} = \frac{1}{a} \arccos\left(\frac{x}{a}\right) + C
\]

\[
\int \frac{dx}{a^2+x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C
\]