4) **Sequences**

- A sequence \( \{a_n\} \) is an ordered list of numbers \( \{a_0, a_1, a_2, a_3, \ldots\} \).

- (The index set can start with \( n = 1 \)).

- Often sequences are given as a **formula**

  \[ a_n = \frac{n}{n^3 + 1} \]

**Examples**

1. \( a_n = n \)
2. \( b_n = \frac{n}{n^3 + 1} \)
3. \( c_n = \cos(\pi n) \)
4. \( g_n = \left(\frac{1}{2}\right)^n \) (Geometric sequence)

L → These are called the **general term** of the sequence.
We can write an sequences' first few terms:

1. \( \{a_n\} = \{0, 1, 2, 3, 4, \ldots \} \)
2. \( \{b_n\} = \{0, \frac{1}{2}, \frac{2}{9}, \frac{3}{28}, \ldots \} \)
3. \( \{c_n\} = \{-1, 1, -1, 1, -1, \ldots \} \)
4. \( \{g_n\} = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \} \)

The sizes of each biirected rectangle slalum, \( \sqrt{1} \) square area = \( \frac{1}{2} \).
Like real functions, sequences have limits:

\[
\lim_{n \to \infty} a_n = \text{the number approached by } a_n \text{ as } n \text{ gets very large}
\]

- When \( \lim_{n \to \infty} a_n \) is finite, \( \{a_n\} \) is called **convergent**.
- We can use what we know about functions to evaluate limits of sequences:

1. If \( a_n = f(n) \) and \( \lim_{x \to \infty} f(x) \) exist,
   
   \[
   \lim_{n \to \infty} a_n = \lim_{x \to \infty} f(x)
   \]

2. If \( f \) is continuous,
   
   \[
   \lim_{n \to \infty} f(a_n) = f\left(\lim_{n \to \infty} a_n\right)
   \]
The limit laws for functions also apply to sequences:

\[
\lim_{n \to \infty} a_n = L, \quad \lim_{n \to \infty} b_n = M
\]

Then:
1. \( \lim_{n \to \infty} (a_n \pm b_n) = L \pm M \)
2. \( \lim_{n \to \infty} a_n b_n = LM \)
3. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{M} \) if \( M \neq 0 \)
4. \( \lim_{n \to \infty} c a_n = cL \) for any constant \( c \)
5. **Squeeze theorem**
   
   If \( a_n \leq b_n \leq c_n \) for all \( n > M \)

   for some \( M \), then

   \[
   \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} b_n \leq \lim_{n \to \infty} c_n
   \]

   and, in particular, if \( \lim_{n \to \infty} a_n = L = \lim_{n \to \infty} c_n \)

   \( \lim_{n \to \infty} b_n = L \)
Examples:

1. \( \lim_{n \to \infty} 10^{-\frac{1}{n}} = 10^{\lim_{n \to \infty} -\frac{1}{n}} \) since the function \( 10^x \) is continuous.

   Now, we know \( \lim_{n \to \infty} \frac{1}{n} = 0 \), hence

   \[ \lim_{n \to \infty} 10^{-\frac{1}{n}} = 10^0 = 1 \]

2. Consider the Cauchy limit \( \lim_{x \to \infty} \frac{x}{\sqrt{x^3 + 1}} \)

   and note that it approaches the indeterminate form \( \frac{\infty}{\infty} \), so we \( \rightarrow \)
We will apply L'Hopital's rule.

First, however, we will use some intuition: since the degree of the bottom is 3/2 and the top is 1, we will assume the limit exists and is

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^3 + 1}} = \text{L}$$

Now apply continuous function $f(x) = x^2$ to both sides:

$$\lim_{x \to \infty} \frac{x^2}{x^3 + 1} = \text{L}^2$$

Now we can see that the left side $\text{L} = 0$, hence $\text{L}^2 = 0$

Therefore,

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^3 + 1}} = 0$$
Two important classes of sequences are bounded and monotonic sequences.

1. Definition: \( \{a_n\} \) is bounded
   - From above if there is some \( M \) so that \( a_n \leq M \) for all \( n \).
   - From below if \( a_n \geq M \) for all \( n \).

2. Definition: \( \{a_n\} \) is monotonic if it is
   - Increasing: \( a_n < a_{n+1} \) for all \( n \)
   - Decreasing: \( a_n > a_{n+1} \) for all \( n \)
Theorem
Convergent sequences are bounded

Theorem
Bounded monotonic sequences are convergent

Example:
\( a_n = 2^{1/n} \)

Consider \( a_{n+1} = 2^{\frac{1}{n+1}} \leq 2^{\frac{1}{n}} = a_n \)

Hence \( \{a_n\} \) is decreasing, and we know \( 2^{1/n} > 0 \), hence bounded below.

So \( \{a_n\} \) is convergent.
**Theorem (Geometric Sequence)**

\[
\lim_{n \to \infty} r a^n = \begin{cases} 
  \text{divergent for } a \leq 1 \\
  0 & \text{for } -1 < a < 1 \\
  1 & \text{for } a = 1 \\
  +\infty & \text{for } a > 1 
\end{cases}
\]

**Examples Cont'd**

4. \( P_n = \frac{2^{n+1}}{3^{n-1}} \)

See that \( P_n = \frac{2^{n+1}}{3^{n-1}} = \frac{2^n(2)}{3^{n}(3^{-1})} = 6 \left(\frac{2}{3}\right)^n \), hence

\[
\lim_{n \to \infty} P_n = 0 \quad \text{since } -1 < \frac{2}{3} < 1
\]