Some Eigenvalue/Eigenvector/Diagonalization Problems
Math 18 – Linear Algebra
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Answers are in red.

For each given matrix $A$, find its eigenvalues, a basis for each eigenspace, and determine whether it is diagonalizable. If it is diagonalizable, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

Exercise 1. $A = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 3 & -1 \\ -2 & 1 & 1 \end{bmatrix}$

• The characteristic polynomial is $-\lambda(\lambda - 2)^2$ so the eigenvalues are $\lambda = 0$ and $\lambda = 2$.
• The eigenspace corresponding to $\lambda = 0$ is the solutions to the homogeneous problem $(A - 0I)x = 0$. Row reducing tells us that $x_3$ is the only free variable and the solutions are of the form:
  $$\begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}.$$ So
  $$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\},$$
is a basis for the 0-eigenspace of $A$. Note that the dimension of the 0-eigenspace is equal to the multiplicity of the eigenvalue $\lambda = 0$.
• The eigenspace corresponding to $\lambda = 2$ is the solutions to the homogeneous problem $(A - 2I)x = 0$. Row reducing tells us that $x_2$ and $x_3$ are free variables and that all solutions are of the form:
  $$\begin{bmatrix} \frac{x_2}{2} - \frac{x_3}{2} \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix},$$
so that:
  $$\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$ Is a basis for the 2-eigenspace of $A$. Note that the dimension of the 2-eigenspace is equal to the multiplicity of the eigenvalue $\lambda = 2$. (What if I told you the answer was
  $$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\},$$
would you be able to tell that the two bases span the same set?)
• Each eigenspace has dimension equal to the multiplicity of its corresponding eigenvalue, so we have “enough” eigenvectors to form a basis for $\mathbb{R}^3$ consisting of eigenvectors of $A$:
  $$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}.$$ And so $A$ is diagonalizable: $A = PDP^{-1}$, where
  $$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$ ($D$ consists of the eigenvalues of $A$ and $P$ consists of the basis vectors we found, arranged as the columns of $P$).

Exercise 2. $A = \begin{bmatrix} -10 & 12 & -30 \\ 0 & 2 & 0 \\ 4 & 0 & 12 \end{bmatrix}$

• The characteristic polynomial of $A$ is $-\lambda(\lambda - 2)^2$. So the eigenvalues are $\lambda = 0$ and $\lambda = 2$.
• The eigenspace corresponding to $\lambda = 0$ is one dimensional; here’s a basis:
  $$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$
• The eigenspace corresponding to $\lambda = 2$ is also one dimensional; here’s a basis:

$$\begin{bmatrix}
-4 \\
0 \\
1
\end{bmatrix}.$$ 

• Because the multiplicity of the eigenvalue $\lambda = 2$ is 2 but its corresponding eigenspace is of dimension one, we don’t have “enough” eigenvectors to span $\mathbb{R}^3$. So this matrix is not diagonalizable.

Exercise 3. $A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 4 \\
-3 & -3 & -1
\end{bmatrix}$

• The characteristic polynomial of $A$ is $-\lambda(\lambda^2 - 5\lambda + 12)$. The only eigenvalue is $\lambda = 0$.
• The eigenspace corresponding to $\lambda = 0$ is one dimensional; here’s a basis:

$$\begin{bmatrix}
7 \\
-8 \\
-3
\end{bmatrix}.$$ 

• $A$ has no eigenvectors other than $\lambda = 0$ and the 0-eigenspace is one dimensional, so it’s impossible to find a basis for $\mathbb{R}^3$ consisting of eigenvectors. So $A$ is not diagonalizable.

Exercise 4. $A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 4 \\
3 & 3 & 1
\end{bmatrix}$

• The characteristic polynomial of $A$ is $-\lambda(\lambda + 2)(\lambda - 9)$. The eigenvalues of $A$ are $\lambda = 0, -2, 9$.
• Note that because each eigenvalue is guaranteed to have at least one (nonzero) eigenvector, and because eigenvectors corresponding to distinct eigenvalues must be linearly independent, $A$ will definitely have 3 linearly independent eigenvectors (one for each eigenvalue). So $A$ is diagonalizable.
• The eigenspace corresponding to $\lambda = 0$ is one dimensional; here’s a basis:

$$\begin{bmatrix}
7 \\
-8 \\
-3
\end{bmatrix}.$$ 

• The eigenspace corresponding to $\lambda = -2$ is one dimensional; here’s a basis:

$$\begin{bmatrix}
1 \\
0 \\
-1
\end{bmatrix}.$$ 

• The eigenspace corresponding to $\lambda = -9$ is one dimensional; here’s a basis:

$$\begin{bmatrix}
5 \\
11 \\
6
\end{bmatrix}.$$ 

• $A = PDP^{-1}$ where:

$$D = \begin{bmatrix}
0 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & 9
\end{bmatrix}, \quad P = \begin{bmatrix}
7 & 1 & 5 \\
-8 & 0 & 11 \\
-3 & -1 & 6
\end{bmatrix}.$$ 

($D$ consists of the eigenvalues of $A$ and $P$ consists of the basis vectors we found, arranged as the columns of $P$).

Exercise 5. Is it possible for a $3 \times 3$ matrix to have no eigenvalues? (Hint: Use Calculus.)

A $3 \times 3$ matrix must have at least one eigenvalue. That’s because its characteristic polynomial $p(\lambda)$ is degree 3. Note that the coefficient of the $\lambda^3$ term is $-1$, so we know $\lim_{\lambda \to -\infty} p(\lambda) = -\infty$ and $\lim_{\lambda \to \infty} p(\lambda) = \infty$ (Why?). Combining this with the fact that all polynomials are continuous functions, the Intermediate Value Theorem tells us there must be a $\lambda$ for which $p(\lambda) = 0$. That $\lambda$ is, by definition, an eigenvalue of $A$.

So $3 \times 3$ matrices, thought of as linear transformations, always have a “stretching” component.

Exercise 6. Is it possible for a $4 \times 4$ matrix to have no eigenvalues? Yes. You can explicitly find a matrix that doesn’t have any eigenvalues. (Hint: If $A$ represents a linear transformation that rotates in the $x_1$ and $x_2$ axes and also rotates in the $x_3$ and $x_4$ axes, then it always takes any vector to a different (non-parallel) vector.)

By the way, notice that the proof for the previous Exercise doesn’t work for this case (Where does it fail?).