**Gram-Schmidt Exercises**

**Math 18 – Linear Algebra**

Last modified Sat, 12/8.

**Exercise 1.** Consider the space of all polynomials \( P \). We define a dot product on \( P \) in the following way:

\[
\langle p, q \rangle = \int_0^1 p(t)q(t) \, dt.
\]

(We use the sharp bracket notation for all dot products to avoid confusion because we’ll also be multiplying functions; so if you see, for example \( t \cdot t^2 \), that’s just the function \( t^3 \); however \( \langle t, t^2 \rangle \) is the dot product of \( t \) and \( t^2 \).) With this dot product, find the vector in \( P_2 \) (thought of as a subspace of \( P \)) that’s closest to \( p(t) = t^3 \).

We first find an orthogonal basis, \( \{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} \), for \( P_2 \) using the Gram-Schmidt process, starting with the “standard basis” \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} = \{1, t, t^2\} \) for \( P_2 \):

1. \( \vec{u}_1 = \vec{v}_1 = 1 \).
2. Now we subtract \( -\vec{v}_2 \) – the part of \( \vec{v}_2 \) that’s in the direction \( \vec{u}_1 \). The result, \( \vec{u}_2 \), will thus be orthogonal to \( \vec{u}_1 \):

\[
\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1} \vec{v}_2 = t - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1.
\]

Thus the result (\( \vec{u}_2 \)) should be orthogonal to \( \vec{u}_1 \). Computing the dot products, we have:

\[
\langle \vec{v}_2, \vec{u}_1 \rangle = \int_0^1 t \cdot 1 \, dt = \frac{1}{2},
\]

\[
\|\vec{u}_1\|^2 = \int_0^1 1 \, dt = 1^2 = 1.
\]

So

\[
\vec{u}_2 = t - \frac{1}{2}.
\]

You check: \( \vec{u}_2 \) is orthogonal to \( \vec{u}_1 \).

3. Now subtract \( -\vec{v}_3 \) – the part of \( \vec{v}_3 \) that are in the directions \( \vec{u}_1 \) and \( \vec{u}_2 \). The result, \( \vec{u}_3 \), will thus be orthogonal to both \( \vec{u}_1 \) and \( \vec{u}_2 \):

\[
\vec{u}_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1} \vec{v}_3 - \text{proj}_{\vec{u}_2} \vec{v}_3 = t - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2.
\]

Computing the dot products:

\[
\langle \vec{v}_3, \vec{u}_1 \rangle = \int_0^1 t^2 \cdot 1 \, dt = \frac{1}{3},
\]

\[
\langle \vec{v}_3, \vec{u}_2 \rangle = \int_0^1 t^2 \cdot (t - \frac{1}{2}) \, dt = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.
\]

\[
\|\vec{u}_2\|^2 = \langle \vec{u}_2, \vec{u}_2 \rangle = \int_0^1 \left( t - \frac{1}{2} \right)^2 \, dt = \frac{1}{24}.
\]

So

\[
\vec{u}_3 = t^3 - \frac{1}{3} \cdot 1 - \left( t - \frac{1}{2} \right) = t^3 - t + \frac{1}{6}.
\]

You check: \( \vec{u}_2 \) is orthogonal to \( \vec{u}_1 \) and \( \vec{u}_2 \).

We now have our orthogonal basis

\[
\{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \} = \{1, t - \frac{1}{2}, t^2 - t + \frac{1}{6} \}.
\]

With this orthogonal basis, we can find the closest point on \( P_2 \) to \( p(t) = t^3 \) (this is the “orthogonal projection of \( p(t) \) onto \( P_2 \)). This is simply the sum of the projections of \( t^3 \) onto each of our basis vectors (note this only works if our basis vectors are orthogonal):

\[
\text{proj}_{\vec{u}_1} t^3 + \text{proj}_{\vec{u}_2} t^3 + \text{proj}_{\vec{u}_3} t^3 = \frac{\langle t^3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \vec{u}_1 + \frac{\langle t^3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \vec{u}_2 + \frac{\langle t^3, \vec{u}_3 \rangle}{\|\vec{u}_3\|^2} \vec{u}_3.
\]

Here is each dot product (that we haven’t already computed):

\[
\langle t^3, \vec{u}_1 \rangle = \int_0^1 t^3 \cdot 1 \, dt = \frac{1}{4},
\]

\[
\langle t^3, \vec{u}_2 \rangle = \int_0^1 t^3 \cdot \left( t - \frac{1}{2} \right) \, dt = \frac{1}{24},
\]

\[
\langle t^3, \vec{u}_3 \rangle = \int_0^1 t^3 \cdot \left( t^2 - t + \frac{1}{6} \right) \, dt = \frac{1}{108}.
\]

\[
\|\vec{u}_3\|^2 = \langle \vec{u}_3, \vec{u}_3 \rangle = \int_0^1 \left( t^2 - t + \frac{1}{6} \right) \left( t^2 - t + \frac{1}{6} \right) \, dt = \frac{1}{108}.
\]
So our solution is:

\[
\text{proj}_{\mathbf{u}_1} t^3 + \text{proj}_{\mathbf{u}_2} t^3 + \text{proj}_{\mathbf{u}_3} t^3 = \frac{1}{4} \mathbf{u}_1 + 12 \frac{1}{80} \mathbf{u}_2 + 180 \frac{1}{120} \mathbf{u}_3
\]

\[
= \frac{1}{4} + \frac{9}{80}(t - \frac{1}{2}) + \frac{3}{2}(t^2 - t + \frac{1}{6})
\]

\[
= \frac{3}{2} t^2 - \frac{3}{5} t + \frac{1}{20}.
\]

**Exercise 2.**

(a) Find an orthonormal basis for \( P_2 \) (thought of as a subspace of \( P \)).

(b) Find the vector in \( P_2 \) that’s closest to \( q(t) = t^4 \).

**Exercise 3.** Define another dot product on \( P \) in the following way:

\[
\langle p, q \rangle = \int_{-2}^{2} p(t) q(t) \, dt.
\]

Find the vector in \( P_2 \) (thought of as a subspace of \( P \)) that’s closest to \( p(t) = t^3 \).