Solutions

Name: ____________________________

Student ID No.: ____________________

Discussion Section: ___________________

Math 18 Midterm I (ver. a)
Fall 2018

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1. (28 Points.) The following are True/False questions. **For this problem only, you do not have to show any work.** There will be no partial credit given for this problem. For this problem:

- A correct answer gives 4 points.
- An incorrect answer gives 0 points.
- If you leave the space blank, you receive 2 points.

**F** (a) Five vectors in $\mathbb{R}^5$ always span $\mathbb{R}^5$.

**F** (b) If $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $U : \mathbb{R}^m \rightarrow \mathbb{R}^p$ are both linear transformations, then the standard matrix for the linear transformation $U \circ T$ is $n \times p$.

**T** (c) If the Row Echelon form of a matrix $A$ has a pivot in every column, then whenever $A\vec{x} = \vec{b}$ has a solution, it must be unique.

**T** (d) The sum of two solutions to the homogeneous problem $A\vec{x} = \vec{0}$ is also a solution.

**T** (e) If a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is both one-to-one and onto, then $n$ must equal $m$.

**F** (f) Two vectors whose entries are all positive must be linearly independent.

**T** (g) Assume $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation and $\vec{v}_1, \ldots, \vec{v}_p$ are vectors in $\mathbb{R}^n$. If $\{T(\vec{v}_1), \ldots, T(\vec{v}_p)\}$ are linearly independent, then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ are also linearly independent.
2. Each statement below is either true (in all cases) or false (in at least one case).
If false, construct a counterexample. If true, give a justification that carefully notes everywhere you use the definition of “linear (in)dependence.”

(a) (12 Points.) If \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent and \( \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \) are linearly dependent, then \( \vec{v}_3 \) is in \( \text{Span} \{ \vec{v}_1, \vec{v}_2 \} \).

(b) (12 Points.) If \( \vec{u}_1 \) and \( \vec{u}_2 \) are linearly dependent vectors in \( \mathbb{R}^2 \), then one of these vectors must be a constant multiple of the other.

(a) True. We want to show that \( \vec{v}_1 \) can be written as a linear combination of \( v_1 \) and \( v_2 \):
\[ \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \text{ being linearly dependent means that there are numbers } a_1, a_2, \text{ and } a_3 \text{ not all zero so that} \]
\[ a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3 = 0. \]
Note that \( a_1 \neq 0 \) because otherwise we would have
\[ a_2 \vec{v}_2 + a_3 \vec{v}_3 = 0, \]
where at least one of \( a_2 \) or \( a_3 \) is nonzero – which can’t happen because \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent.
So we’re safe to divide through by \( a_1 \), giving:
\[ \vec{v}_1 + \frac{a_2}{a_1} \vec{v}_2 + \frac{a_3}{a_1} \vec{v}_3 = 0, \]
and thus:
\[ \vec{v}_1 = -\frac{a_2}{a_1} \vec{v}_2 - \frac{a_3}{a_1} \vec{v}_3; \]
and so \( \vec{v}_1 \) is in \( \text{Span} \{ \vec{v}_1, \vec{v}_2 \} \).

(b) True. Because \( u_1 \) and \( u_2 \) are linearly dependent, there are \( c_1 \) and \( c_2 \) not both zero so that
\[ c_1 \vec{u}_1 + c_2 \vec{u}_2 = 0. \]
If \( c_1 \neq 0 \), then
\[ \vec{u}_1 = -\frac{c_2}{c_1} \vec{u}_2. \]
So \( u_1 \) is a constant multiple of \( u_2 \). If, on the other hand, \( c_1 = 0 \), then \( c_2 \neq 0 \), in which case:
\[ \vec{u}_2 = -\frac{c_1}{c_2} \vec{u}_1, \]
and thus \( u_2 \) is a constant multiple of \( u_1 \).
3. Consider the Linear Transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ whose standard matrix is:

$$
\begin{bmatrix}
1 & -2 & 5 & -3 \\
2 & 3 & 3 & 1 \\
4 & 1 & 11 & -3
\end{bmatrix}.
$$

(a) (18 Points.) Is $T$ onto?

(b) (6 Points.) If $T$ is onto, solve $T(\vec{x}) = \vec{y}$ for a nonzero vector $\vec{y}$ of your choice. If it is not onto, find a vector $\vec{y}$ for which $\vec{y}$ is not in the Range of $T$.

Reminder: you must show work to receive credit.

Let's solve this entire problem in one computation (you don't have to, but it's more efficient to do so). In particular, we solve, for any $\vec{b}$:

$$
\begin{bmatrix}
1 & -2 & 5 & -3 \\
2 & 3 & 3 & 1 \\
4 & 1 & 11 & -3
\end{bmatrix}
\begin{bmatrix}
\vec{x} \\
\vec{b}_1 \\
\vec{b}_2 \\
\vec{b}_3
\end{bmatrix}.
$$

Row reduce:

$$
\begin{bmatrix}
1 & -2 & 5 & -3 \\
2 & 3 & 3 & 1 \\
4 & 1 & 11 & -3
\end{bmatrix}
\xrightarrow{R_2=-2R_1+R_2}
\begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 7 & -7 & 7 \\
4 & 1 & 11 & -3
\end{bmatrix}
\xrightarrow{R_3=-4R_1+R_3}
\begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 7 & -7 & 7 \\
0 & 9 & -9 & 9
\end{bmatrix}
\xrightarrow{R_3=-9R_2+R_3}
\begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 7 & -7 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{R_3=-10R_1+R_3}
\begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 7 & -7 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{R_3=-10R_1+R_3}
\begin{bmatrix}
1 & -2 & 5 & -3 \\
0 & 7 & -7 & 7 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

We can now see that as long as we choose $b_1$, $b_2$, and $b_3$ so that $-10b_1 - 9b_2 + b_3 \neq 0$, then there is no solution to this system of equations. This means we've found a vector that is not the image of any vector under the transformation $T$. For example,

$$
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
$$

will work.
4. (24 Points.) Consider the following system of equations (where \( a \) and \( b \) are constants):

\[
\begin{align*}
-2x_1 + 4x_2 &= 1 \\
4x_1 - ax_2 &= b
\end{align*}
\]

(a) Choose \( a \) and \( b \) so that the system has a unique solution.

(b) Choose \( a \) and \( b \) so that the system has no solutions.

(c) Choose \( a \) and \( b \) so that the system has many solutions.

Let's row reduce:

\[
\begin{bmatrix}
-2 & 4 & 1 \\
4 & -a & b \\
\end{bmatrix}
\xrightarrow{R_2 = 2R_1 + R_2}
\begin{bmatrix}
-2 & 4 & 1 \\
0 & 8-a & 2+b \\
\end{bmatrix}.
\]

(a) For a unique solution, choose any \( a \) and \( b \) so that both the entries that contain \( a \) and \( b \) are non-zero (so that our system has 2 pivots); \( a = b = 0 \) will work.

(b) For no solutions, choose \( a = 8 \) and any \( b \) such that \( 2 + b \neq 0 \); \( b = 0 \) will work (the second equation would then be \( 0x_1 + 0x_2 = 2 \), which has no solution).

(c) For many solutions, choose \( a = 8 \) and \( b = -2 \), making the final row zero, thus giving us one free variable for this system.