Solutions

Name: ________________________________

Student ID No.: ________________________

Discussion Section: ______________________

Math 20E Final Exam (ver. c)
Fall 2018

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1. (24 Points.) For each vector field, calculate the work done in moving a particle along the given path:

(a) (12 Points.)
- \( \vec{F} = xy \vec{i} + x \vec{j} - z \vec{k} \).
- The path is the straight line segment from \((1,2,-1)\) to \((3,1,2)\).

(b) (12 Points.)
- \( \vec{G} = 2xe^{x^2}y\vec{i} + (e^{x^2} + z) \vec{j} + (y + 2z) \vec{k} \).
- The path is the one formed by traveling in straight lines from \((1,2,4)\) to \((2,7,5)\), and finally to \((0,3,1)\).

(a) This is a straight line segment pointing in the direction \((3 - 1, 1 - 2, 2 - (-1)) = (2, -1, 3)\). We parametrize with:

\[ c(t) = (1 + 2t, 2 - t, -1 + 3t), \]

where \(0 \leq t \leq 1\). Its velocity is:

\[ c'(t) = 2\vec{i} - \vec{j} + 3\vec{k}. \]

So the work is:

\[ \int \vec{F} \cdot d\vec{s} = \int_{t=0}^{t=1} \vec{F} \cdot c'(t) \, dt \]
\[ = \int_{t=0}^{t=1} 2xy - x - 3z \, dt \]
\[ = \int_{t=0}^{t=1} 2(1 + 2t)(2 - t) - (1 + 2t) - 3(-1 + 3t) \, dt \]
\[ = \int_{t=0}^{t=1} -4t^2 - 5t + 6 \, dt \]
\[ = \frac{13}{6}. \]

(b) This is a conservative vector field because \( \text{curl} \vec{G} = \vec{0} \) (you check). We find \( f \) so that \( \nabla f = \vec{G} \), then our integral becomes easy:

\( f_x \) must match the \( \vec{i} \) term of \( \vec{G} \), so \( f_x(x,y,z) = 2xe^{x^2}y \). Integrate both sides of this equation with respect to \( x \) to obtain

\[ f(x,y,z) = xe^{x^2}y + g(y,z), \]

for some function \( g(y,z) \). Differentiate this with respect to \( y \) and match this with the \( \vec{j} \) term of \( \vec{G} \), giving:

\[ e^{x^2}g_y(y,z) = fy(x,y,z) = e^{x^2} + z. \]

So \( g_y(y,z) = z \), which means \( g(y,z) = yz + h(z) \) for some function \( h(z) \). And so updating our \( f \):

\[ f(x,y,z) = xe^{x^2}y + yz + h(z). \]

Differentiate this with respect to \( z \) and match this with the \( \vec{k} \) term of \( \vec{G} \), giving:

\[ y + h'(z) = fz(x,y,z) = y + 2z. \]

So \( h'(z) = 2z \), so \( h(z) = z \) (no need to add a constant because we just need one potential function). We end up with:

\[ f(x,y,z) = xe^{x^2}y + yz + z^2. \]

Thus our integral is:

\[ \int_c \vec{G} \cdot d\vec{s} = \int_c \nabla f \cdot d\vec{s} = f(\text{end}) - f(\text{start}) = f(0,3,1) - f(1,2,4) = -2e - 17. \]
2. (24 Points.) For each vector field, calculate the work done in moving a particle along the given path:

(a) (12 Points.)
- \( \vec{F} = 3x^2 e^{x^3} \hat{i} + e^{x^3} \hat{j} + (y + z) \hat{k} \).
- The path is the one formed by traveling in straight lines from (2, 0, 0) to (0, 0, 6) to (0, 3, 0), and back to (2, 0, 0).

(b) (12 Points.) Note: this force field and path are in \( \mathbb{R}^2 \).
- \( \vec{G} = \left( (\ln x)^3 - y \right) \hat{i} + \left( e^{x^3} + x \right) \hat{j} \)
- The path is formed by traveling in straight lines from (1, 3) to (4, 7) to (4, 1), and back to (1, 3).

(a) This line integral is over a closed curve so we have the option to use Stokes’ Theorem. The line integral looks hard but our path is the boundary of a triangular (planar) region \( S \) (easy to work with), and \( \text{curl} \vec{F} = \vec{v} \) (very simple), so we opt to use Stokes’ Theorem, which means we need to parametrize our triangular region.

In order to parametrize the plane, we need its normal: if we call \( P = (2,0,0), Q = (0,0,6) \), and \( R = (0,3,0) \), then the normal direction is \( \vec{PQ} \times \vec{PR} = -18\hat{j} - 12\hat{k} - 6\hat{i} \). So the equation of the plane is: \(-18(x - 2) - 12y - 6z = 0\). We parametrize it by thinking of it as a graph: \((u,v) \mapsto (u, v, -3(u-2) - 2v)\). The region corresponding to \( S \) in the \( uv\)-plane is the triangle \( T \) bounded by \((2,0), (0,3), \) and \((0,0)\). So our integral – noting that our parametrization is in the wrong orientation – is

\[
\int_{c} \vec{F} \cdot d\vec{s} = - \iint_{S} \text{curl} \vec{F} \cdot d\vec{s},
\]

\[
= - \iint_{T} \text{curl} \vec{F} \cdot ||\vec{T}_{u} \times \vec{T}_{v}|| \, du \, dv
\]

\[
= - \iint_{T} \vec{v} \cdot (3\hat{i} + 2\hat{j} + \hat{k}) \, du \, dv
\]

\[
= - \iint_{T} 3 \, du \, dv
\]

\[
= -3 \cdot \text{(Area of triangle)}
\]

\[
= -9
\]

Alternatively, if you’re feeling fancy: once we know the normal to \( S \), we have its unit normal: \( \vec{n} = -\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}) \)
(note this is in the correct orientation). Then we can compute:

\[
\int_{c} \vec{F} \cdot d\vec{s} = \iint_{S} \text{curl} \vec{F} \cdot d\vec{s},
\]

\[
= \iint_{S} \text{curl} \vec{F} \cdot \vec{n} \, dS,
\]

\[
= \iint_{S} \vec{v} \cdot \left( -\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}) \right) \, dS,
\]

\[
= \iint_{S} -\frac{3}{\sqrt{14}} \, dS,
\]

\[
= -\frac{3}{\sqrt{14}} \cdot \text{(Area of S)},
\]

\[
= -\frac{3}{\sqrt{14}} \cdot \frac{1}{2} \cdot ||\vec{PQ} \times \vec{PR}||,
\]

\[
= -\frac{3}{\sqrt{14}} \cdot \frac{1}{2} \cdot (6 \sqrt{14})
\]

\[
= -9.
\]

(b) Our path, call it \( c \), is a closed curve so we have the option of using Green’s Theorem. The line integrals look hard but \( \text{curl} \vec{G} = 2 \) (this is the scalar curl) is simple; also, the region enclosed by our curve is a triangle. So we opt to use Green’s Theorem (note our path’s orientation requires a minus sign in front of the double integral):

\[
\int_{c} \vec{G} \cdot d\vec{s} = - \iint_{S} \text{curl} \vec{G} \, du \, dv = - \iint_{S} 2 \, du \, dv = -2 \cdot \text{(Area of triangle)} = -18.
\]
3. (26 Points.) Consider the region \( W \), in \( \mathbb{R}^3 \), that lies below the paraboloid \( z = 9 - x^2 - y^2 \) and above the cone \( z = \sqrt{x^2 + y^2} - 3 \). Also let \( \vec{F} = \vec{F} \). Compute both sides of the divergence theorem for this region.

The Divergence Theorem says that

\[
\iiint_W \text{div} \vec{F} \, dV = \iiint_{\partial W} \vec{F} \cdot d\vec{S}.
\]

- First we compute the left side of the Divergence Theorem.

\[
\iiint_W \text{div} \vec{F} \, dV = \iiint_W 1 \, dV = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} \int_{z=r^2}^{z=9-r^2} r \, dz \, dr \, d\theta
\]

\[
= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} \left( (9-r^2) - (r-3) \right) r \, dz \, dr \, d\theta
\]

\[
= \frac{81\pi}{2}.
\]

- Now for the right hand side. We call \( P \) the portion of \( \partial W \) that’s on the paraboloid, and \( C \) the portion of \( \partial W \) that’s on the cone.

For the paraboloid, we parametrize with \((u,v) \mapsto (u,v,9 - u^2 - v^2)\). Then \( T_u = (1,0,0) \), \( T_v = (0,1,0) \), and so \( T_u \times T_v = 2u\hat{i} + 2v\hat{j} + \hat{k} \). So (noting that our parametrization has the correct orientation) and letting \( D \) be the circle of radius 3 about the origin (in the uv-plane):

\[
\int_P \vec{F} \cdot d\vec{S} = \int_D z \hat{k} \cdot \left( 2u\hat{i} + 2v\hat{j} + \hat{k} \right) \, dudv
\]

\[
= \int_D z \, dudv
\]

\[
= \int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} \left( (9-r^2) - (r-3) \right) r \, d\theta \, dr
\]

\[
= \frac{81\pi}{2}.
\]

For the cone, we parametrize with \((u,v) \mapsto (u,v,\sqrt{u^2+v^2} - 3)\). Then \( T_u = \frac{u}{\sqrt{u^2+v^2}} \hat{i} + \frac{v}{\sqrt{u^2+v^2}} \hat{j} + \hat{k} \), and so \( T_u \times T_v = \frac{u}{\sqrt{u^2+v^2}} \hat{i} + \frac{v}{\sqrt{u^2+v^2}} \hat{j} + \hat{k} \). So (noting that our parametrization has the wrong orientation):

\[
\int_C \vec{F} \cdot d\vec{S} = -\int_D z \hat{k} \cdot \left( \frac{u}{\sqrt{u^2+v^2}} \hat{i} + \frac{v}{\sqrt{u^2+v^2}} \hat{j} + \hat{k} \right) \, dudv
\]

\[
= -\int_D z \, dudv
\]

\[
= -\int_{r=0}^{r=3} \int_{\theta=0}^{\theta=2\pi} \left( (9-r^2) - (r-3) \right) r \, d\theta \, dr
\]

\[
= 9\pi.
\]

We conclude

\[
\iiint_{\partial W} \vec{F} \cdot d\vec{S} = \int_P \vec{F} \cdot d\vec{S} + \int_C \vec{F} \cdot d\vec{S} = \left( \frac{81}{2} + 9 \right) \pi = \frac{99\pi}{2}.
\]
4. (25 Points.) Let \( S \) be the part of the surface

\[
\sqrt{x^2 + y^2} = \frac{1}{4} z^2 + 1
\]

that lies between \( z = -1 \) and \( z = 2 \). Compute the integral:

\[
\iint_S xy^2 z \, dS.
\]

This surface is rotationally symmetric about the \( z \)-axis. In cylindrical coordinates, it’s \( r = \frac{1}{4} z^2 + 1 \), which is a “sideways” parabola rotated about the \( z \)-axis.

To parametrize, we first think of \( r = \frac{1}{4} z^2 + 1 \) as a curve in the \( rz \)-plane. We parametrize that curve using parameter \( u \):

\[
\begin{align*}
  z &= u \\
  r &= \frac{1}{4} u^2 + 1
\end{align*}
\]

Now add a second parameter, \( v \), that represents rotation about the \( z \)-axis:

\[
\begin{align*}
  z &= u \\
  r &= \frac{1}{4} u^2 + 1 \\
  \theta &= v
\end{align*}
\]

We translate this parametrization to rectangular coordinates using:

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= z
\end{align*}
\]

Now compute:

\[
\vec{T}_u = \left( \frac{1}{2} u \cos v \right) \hat{i} + \left( \frac{1}{2} u \sin v \right) \hat{j} + \hat{k}
\]

\[
\vec{T}_u = - \left[ \left( \frac{1}{4} u^2 + 1 \right) \sin v \right] \hat{i} + \left[ \left( \frac{1}{4} u^2 + 1 \right) \cos v \right] \hat{j}
\]

\[
\vec{T}_u \times \vec{T}_v = - \left[ \left( \frac{1}{4} u^2 + 1 \right) \cos v \right] \hat{i} - \left[ \left( \frac{1}{4} u^2 + 1 \right) \sin v \right] \hat{j} + \frac{1}{2} u \left( \frac{1}{4} u^2 + 1 \right) \hat{k}
\]

So

\[
||\vec{T}_u \times \vec{T}_v|| = \sqrt{\left( \frac{1}{4} u^2 + 1 \right)^2 + \frac{1}{2} u^2 \left( \frac{1}{4} u^2 + 1 \right)^2}
\]

\[
= \left( \frac{1}{4} u^2 + 1 \right) \sqrt{1 + \frac{1}{4} u^2}
\]

\[
= \left( \frac{1}{4} u^2 + 1 \right)^{\frac{3}{2}}.
\]

Now we can integrate:

\[
\iint_S xy^2 z \, dS = \int_{\theta=0}^{\theta=2\pi} \int_{u=-1}^{u=2} \left( \frac{1}{4} u^2 + 1 \right) \cos v \cdot \left( \frac{1}{4} u^2 + 1 \right) \frac{1}{2} \sin^2 v \cdot u \cdot \left( \frac{1}{4} u^2 + 1 \right)^{\frac{1}{2}} \, du \, dv
\]

\[
= \int_{\theta=0}^{\theta=2\pi} \int_{u=-1}^{u=2} u \left( \frac{1}{4} u^2 + 1 \right)^{\frac{3}{2}} \cos v \sin^2 v \, du \, dv
\]

\[
= 0.
\]
5. (25 Points.)

(a) (18 Points.) Let $S_1$ be the part of the cylinder $x^2 + y^2 = 16$ that lies between the planes $z = -3$ and $z = 3$. If $\vec{F} = (2yz + e^{x^2}) \vec{i} + x \vec{j} + (xy + z) \vec{k}$, find

$$\iint_{S_1} \text{curl} \vec{F} \cdot d\vec{S},$$

where $\vec{n}$ points radially outward, away from the $z$-axis.

(b) (7 Points.) Let $S_2$ be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies between the planes $z = -3$ and $z = 3$. For the same $\vec{F}$ as in part (a), find

$$\iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S},$$

where $\vec{n}$ points inward, towards the origin.

(a) The cylinder is rotationally symmetric about the $z$-axis and can be expressed in terms of only $r$‘s and $z$‘s as: $r = 4$.

Using the same parametrization procedure as the previous problem ($r = 4$, $z = u$ to parametrize the curve; then add $\theta = v$ for rotation about the $z$-axis; then convert to rectangular coordinates using $x = r \cos \theta$, $y = r \sin \theta$, $z = z$) our parametrization is:

$$x = 4 \cos v \quad y = 4 \sin v \quad z = u$$

Note that we’re interested in the portion of the cylinder represented by $(u, v) \in [-3, 3] \times [0, 2\pi]$.

Now we can compute:

$$\vec{T}_u = \vec{k} \quad \vec{T}_v = -4 \sin v \vec{i} + 4 \cos v \vec{j}$$

$$\vec{T}_u \times \vec{T}_v = -4 \cos v \vec{i} - 4 \sin v \vec{j}.$$  

Noting that our normal direction is in the wrong orientation:

$$\iint_{S_1} \text{curl} \vec{F} \cdot d\vec{S} = -\int_{v=0}^{v=2\pi} \int_{u=-3}^{u=3} \left( x \vec{i} + y \vec{j} + (1 - 2z) \vec{k} \right) \cdot (-4 \cos v \vec{i} - 4 \sin v \vec{j}) \, du \, dv$$

$$= -\int_{v=0}^{v=2\pi} \int_{u=-3}^{u=3} -4x \cos v - 4y \sin v \, du \, dv$$

$$= -\int_{v=0}^{v=2\pi} \int_{u=-3}^{u=3} -16 \cos^2 v - 16 \sin^2 v \, du \, dv$$

$$= -\int_{v=0}^{v=2\pi} \int_{u=-3}^{u=3} -16 \, du \, dv$$

$$= 16 \cdot 6 \cdot 2\pi$$

$$= 192\pi.$$  

(b) Consider the surface $S$ made up of $S_1$ and $S_2$ put together. Notice that it is a closed surface with no boundary. Orient $S$ using inward pointing normal vectors. Notice that this orientation is consistent with the given orientations for $S_1$ and $S_2$. Because $S$ has no boundary, Stokes’ Theorem tells us:

$$\iint_S \text{curl} \vec{F} \cdot d\vec{S} = 0 = \iint_{S_1} \text{curl} \vec{F} \cdot d\vec{S} + \iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S}.$$  

Thus, we conclude

$$\iint_{S_2} \text{curl} \vec{F} \cdot d\vec{S} = -\iint_{S_1} \text{curl} \vec{F} \cdot d\vec{S} = -192\pi.$$
6. (24 Points.)

(a) (12 Points.) Let \( D^* = [0,2] \times [0,1] \) and define \( T : D^* \to \mathbb{R}^2 \) by \( T(u,v) = (u,-v^2 + 3v) \). Is \( T \) one-to-one? Justify your answer.

(b) (12 Points.) Is the mapping \( U : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( U(u,v) = (uv,u+v) \) onto?

(a) Let's start with \( T(u_1,v_1) = T(u_2,v_2) \). If this implies that \( (u_1,v_1) = (u_2,v_2) \) then the mapping is one-to-one. We have:

\[
(u_1,-v_1^2 + 3v_1) = T(u_1,v_1) = T(u_2,v_2) = (u_2,-v_2^2 + 3v_2).
\]

So we immediately conclude \( u_1 = u_2 \). We also have

\[
-v_1^2 + 3v_1 = -v_2^2 + 3v_2.
\]

Throwing everything to the right hand side, we have

\[
0 = v_1^2 - 3v_1 - v_2^2 + 3v_2 = v_1^2 - v_2^2 - 3(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2) - 3(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2 - 3).
\]

But \( v_1 + v_2 - 3 \) can never be equal to zero since \( D^* = [0,2] \times [0,1] \), which means both \( v_1 \) and \( v_2 \) must be less than 1. Therefore, \( v_1 - v_2 = 0 \), which means \( v_1 = v_2 \).

Alternatively, solve \( T(u,v) = (x,y) \) for \( (u,v) \) in terms of \( x \) and \( y \) explicitly. You'll get \( u = x \) and

\[
v = \frac{-9 \pm \sqrt{81 - 4y}}{2}.
\]

The solution with the plus-sign in front of the root is not viable because:

\[
v = \frac{9 + \sqrt{81 - 4y}}{2} \geq \frac{9}{2} > 1,
\]

which would be outside our legal range of \([0,1]\) for \( v \). Therefore \( T(u,v) = (x,y) \) can have at most one solution, making it one-to-one.

Alternatively, you can say \(-v^2 + 3v\) is an increasing function (take its derivative) on \([0,1]\) so it must be one-to-one there (officially, you'd quote the Mean Value Theorem).

(b) We solve \( U(u,v) = (x,y) \) for \( (u,v) \) in terms of \( x \) and \( y \) and see if there is always a solution for every choice of \( x \) and \( y \). If so, then \( U \) is onto, otherwise, it's not. This means we need to solve the (nonlinear) system:

\[
\begin{align*}
uv &= x \\
 u + v &= y
\end{align*}
\]

for \( u \) and \( v \). The second equation tells us \( u = y - v \). Plugging this into the first equation gives \((y-v)v = x\), which is the same as \( v^2 - vy + x = 0 \). This is a quadratic equation with variable \( v \). We know quadratic equations often have no solutions so this mapping is likely not onto.

To confirm this, we notice that if \( y = 0 \) and \( x = 1 \) the quadratic equation becomes \( v^2 + 1 = 0 \), which has no solution. Therefore our system has no solutions, and thus \( U \) does not map any value to \((1,0)\), so it's not onto.
7. (26 Points.)

- Let \( T(u, v) = (x(u, v), y(u, v)) \) be the mapping defined by \( T(u, v) = (u, u^2 + 3v) \).
- Let \( D^* \) be the triangle bounded by the three lines \( u = 1 \), \( v = 0 \), and \( v = u \).
- Let \( D = T(D^*) \) (the image of \( D^* \) under the mapping \( T \)).
- Let \( f(x, y) = \sqrt{xy} \) be a function defined on \( D \).

The Change of Variables/Coordinates Theorem tells us the integral of \( f \) over the region \( D \) is equal to an integral over the region \( D^* \). Set up, but do not compute the integrals on both sides of the equality.

The Change of Variables Theorem:

\[
\int \int_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv = \int \int_{D} f(x, y) \, dx \, dy.
\]

- For the integral on the left hand side,
  \[
f(x(u, v), y(u, v)) = f(u, u^2 + 3v) = \sqrt{u(u^2 + 3v)}.
\]

The Jacobian term:

\[
\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \right| = \left| \det \begin{pmatrix} 1 & 0 \\ 2u & 3 \end{pmatrix} \right| = 3.
\]

So the integral on the left hand side is:

\[
\int_{u=0}^{u=1} \int_{v=0}^{v=u} \sqrt{u(u^2 + 3v)} \cdot 3 \, du \, dv.
\]

- For the right hand side, we need \( D \). We parametrize each boundary segment of \( D^* \):
  
  - For \( v = 0 \), we use \( t \mapsto (t, 0) \), where \( 0 \leq t \leq 1 \). Applying our mapping to this gives \( (t, t^2) \), which is the parabola \( y = x^2 \). Note that we’re interested in the portion between \((0,0)\) (corresponding to \( t = 0 \)) and \((1,1)\) (corresponding to \( t = 1 \)).
  
  - For \( u = 1 \), we use \( t \mapsto (1, t) \), where \( 0 \leq t \leq 1 \). Applying \( T \), we get \( T(1, t) = (1, 1 + 3t) \). This is the vertical line at \( x = 1 \), starting \((t = 0)\) at \((1,1)\) and ending \((t = 1)\) at \((1,4)\).
  
  - For \( v = u \), we use \( t \mapsto (t, t) \), where \( 0 \leq t \leq 1 \). We have \( T(t, t) = (t, t^2 + 3t) \). So we want the segment of the parabola \( y = x^2 + 3x \) that lies between \((0,0)\) and \((1,4)\).

So \( D \) is the region in the first quadrant bounded below by \( y = x^2 \), above by \( y = x^2 + 3x \), and to the right by \( y = 1 \). Thus, the integral on the right hand side is:

\[
\int_{x=0}^{x=1} \int_{y=x^2}^{y=x^2+3x} \sqrt{xy} \, dx \, dy.
\]
8. (26 Points.)

(a) (6 Points.) Parametrize the curve formed by the intersection of the cylinder \((x-5)^2 + (z-3)^2 = 4\) with the \(xz\)-plane. (Be careful: it’s a “\(z\)” in the equation of the cylinder, not a “\(y\).”)

(b) (14 Points.) Consider the surface \(S\) created by rotating the circle from part (a) about the \(z\)-axis (it looks like a donut). Parametrize this surface.

(c) (6 Points.) Using your parametrization from part (b), find the equation for the tangent plane of \(S\) at the point \((\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 1)\).

(a) This is a circle about the point \((5,0,3)\) with radius 2. Here is a parametrization of a circle of radius 2, in the \(xz\)-plane, about the origin: \(t \mapsto (2 \cos t, 0, 2 \sin t)\). We shift 5 steps in the \(x\) direction and 3 steps in the \(z\) direction to get the circle we want (for \(t \in [0, 2\pi]\)):

\[
\begin{align*}
\ x &= 2 \cos t + 5 \\
\ z &= 2 \sin t + 3 \\
\ y &= 0
\end{align*}
\]

Note that we can always check our answer by plugging in \(x(t) = 2 \cos t + 5 \) and \(y(t) = 2 \sin t + 3\) into \((x-5)^2 + (z-3)^2 = 4\) and confirming that the equation is satisfied.

(b) We start with our parametrization of the circle, this time thought of as being in the \(rz\)-plane (and with parameter \(u\)):

\[
\begin{align*}
\ r &= 2 \cos u + 5 \\
\ z &= 2 \sin u + 3
\end{align*}
\]

We add \(\theta = v\):

\[
\begin{align*}
\ r &= 2 \cos u + 5 \\
\ z &= 2 \sin u + 3 \\
\ \theta &= \nu
\end{align*}
\]

and then translate this to rectangular coordinates using \(x = r \cos \theta, \ y = r \sin \theta, \ z = z\); thus obtaining our parametrization:

\[
\begin{align*}
\ x &= (2 \cos u + 5) \cos v \\
\ y &= (2 \cos u + 5) \sin v \\
\ z &= 2 \sin u + 3
\end{align*}
\]

where \((u,v) \in [0, 2\pi] \times [0, 2\pi]\).

(c) We first find the point in the \(uv\)-plane that corresponds to \(\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 1\right)\). At this point, \(z = 1\), and our parametrization has \(z = 2 \sin u + 3\), so

\[1 = 2 \sin u + 3,
\]

which means \(\sin u = -1\), so \(u = \frac{3\pi}{2}\).

Now for \(v\): our point has \(x\)-coordinate \(x = \frac{5}{\sqrt{2}}\) and our parametrization has \(x = (2 \cos u + 5) \cos v\); plugging in our value for \(u\):

\[x = (2 \cos u + 5) \cos v = (2 \cos \frac{3\pi}{2} + 5) \cos v = 5 \cos v.
\]

So

\[\frac{5}{\sqrt{2}} = 5 \cos v,
\]

which tells us \(v = \frac{\pi}{4}\).

We conclude that \(\left(\frac{3\pi}{2}, \frac{\pi}{4}\right)\) maps to the point \(\left(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}}, 1\right)\).

Now for the tangent plane at \((u,v)\):

\[
\begin{align*}
\vec{T}_u &= -(2 \sin u \cos v) \hat{i} - (2 \sin u \sin v) \hat{j} + 2 \cos u \hat{k} \\
\vec{T}_v &= -(2 \cos u + 5) \sin v \hat{i} + (2 \cos u + 5) \cos v \hat{j} \\
\vec{T}_u \times \vec{T}_v &= -2 \cos u (2 \cos u + 5) \cos v \hat{i} + 2 \cos u (2 \cos u + 5) \sin v \hat{j} - 2 \sin u (2 \cos u + 5) \hat{k} \\
\vec{T}_u \times \vec{T}_v &= 10 \hat{k} \text{ evaluating at } (u,v) = \left(\frac{3\pi}{2}, \frac{\pi}{4}\right)
\end{align*}
\]

So our tangent plane is:

\[10(z - 1) = 0.
\]