<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/25</td>
</tr>
<tr>
<td>2</td>
<td>/26</td>
</tr>
<tr>
<td>3</td>
<td>/24</td>
</tr>
<tr>
<td>4</td>
<td>/25</td>
</tr>
<tr>
<td>Total</td>
<td>/100</td>
</tr>
</tbody>
</table>
1. (25 Points.) Evaluate the integral

$$
\iiint_W 1 \, dx \, dy \, dz
$$

where $W$ is the region determined by the conditions $1 \leq z \leq 5$ and $x^2 + y^2 + z^2 \leq 25$.

$W$ is the region of the sphere (of radius 5 about the origin) that lies above $z = 1$. We integrate using cylindrical coordinates.

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{24}} \int_{z=1}^{z=\sqrt{25-r^2}} 1 \cdot r \, dz \, dr \, d\theta.$$ 

(The innermost integral goes from the plane to the sphere. The outer two integrals is over the disk of radius $\sqrt{24}$ about the origin.) This is equal to

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\sqrt{24}} r \sqrt{25-r^2} \, dr \, d\theta = \int_{\theta=0}^{\theta=2\pi} \left( -\frac{1}{2} \frac{1}{3} (25-r^2)^{\frac{3}{2}} \right)_{r=0}^{r=\sqrt{24}} d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{88}{3} d\theta = \frac{88}{3} \cdot 2\pi = \frac{176}{3} \pi.$$
2. (26 Points.)

(a) (13 Points.) Is the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(u,v) = (2u - 3v, 4u + v)$ onto? Justify your answer.

(b) (13 Points.) Let $D^* = [0,2] \times [0,1]$ and define $U : D^* \to \mathbb{R}^2$ by $U(u,v) = (u,-v^2 + 9v)$. Is $U$ one-to-one? Justify your answer.

(a) This is a linear mapping corresponding to the linear mapping with determinant

$$\det \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} = 2 + 12 = 14 \neq 0.$$ 

Therefore it is onto (and one-to-one as well).

Without appealing to Linear Algebra: We need to show $T(u,v) = (x,y)$ has a solution for all $(x,y) \in \mathbb{R}^2$. So we have to show that the system (with unknowns $u$ and $v$):

$$2u - 3v = x$$
$$4u + v = y$$

is solvable for all $x$ and $y$. We do this by explicitly finding the solutions.

The second equation tells us $v = y - 4u$. Plugging this into the first equation gives $2u - 3(y - 4u) = x$, so that $14u - 3y = x$, giving us $u = \frac{1}{14} x + \frac{3}{14} y$. Therefore $v = y - \frac{2}{7} x - \frac{6}{7} y = -\frac{2}{7} x + \frac{4}{7} y$. And so we conclude

$$T \left( \frac{1}{14} x + \frac{3}{14} y, -\frac{2}{7} x + \frac{4}{7} y \right) = (x,y).$$

This is valid for all choices of $x$ and $y$, telling us $T$ is onto.

(b) There are many ways to do this one.

- Let's start with $U(u_1,v_1) = U(u_2,v_2)$. If this implies that $(u_1,v_1) = (u_2,v_2)$ then the mapping is one-to-one. We have:

$$(u_1,-v_1^2 + 9v_1) = U(u_1,v_1) = U(u_2,v_2) = (u_2,-v_2^2 + 9v_2).$$

So we immediately conclude $u_1 = u_2$. We also have

$$-v_1^2 + 9v_1 = -v_2^2 + 9v_2.$$ 

Throwing everything to the right hand side, we have

$$0 = v_1^2 - 9v_1 - v_2^2 + 9v_2 = v_1^2 - v_2^2 - 9(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2) - 9(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2 - 9).$$

But $v_1 + v_2 - 9$ can never be equal to zero since $D^* = [0,2] \times [0,1]$, which means both $v_1$ and $v_2$ must be less than 1. Therefore, $v_1 - v_2 = 0$, which means $v_1 = v_2$.

- Alternatively, solve for $U(u,v) = (x,y)$ explicitly. You'll get $u = x$ and

$$v = -\frac{9 \pm \sqrt{81 - 4y}}{2}.$$ 

The solution with the plus-sign in front of the root is not viable because:

$$v = \frac{9 + \sqrt{81 - 4y}}{2} \geq \frac{9}{2} > 1,$$

which would be outside our legal range of $[0,1]$ for $v$. Therefore our equation can have at most one solution, making it one-to-one.

- Alternatively, you can say $-v^2 + 9v$ is an increasing function (take its derivative) on $[0,1]$ so it must be one-to-one there (officially, you'd quote the Mean Value Theorem).
3. (24 Points.)

- Let \( T(u,v) = (x(u,v), y(u,v)) \) be the mapping defined by \( T(u,v) = (u,u^2 + 2v) \).
- Let \( D^* \) be the triangle bounded by the three lines \( u = 1, v = 0, \) and \( v = u \).
- Let \( D = T(D^*) \) (the image of \( D^* \) under the mapping \( T \)).
- Let \( f(x,y) = \sqrt{xy} \) be a function defined on \( D \).

The Change of Variables/Coordinates Theorem tells us the integral of \( f \) over the region \( D \) is equal to an integral over the region \( D^* \).

(a) (12 Points.) Set up, but do not compute the integral on the left side of the equality.

(b) (12 Points.) Set up, but do not compute the integral on the right side of the equality.

Your integrals should have complete limits of integration; your integrands should be explicitly expressed in terms of the variables of integration.

The Change of Variables Theorem:

\[
\iint_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv = \iint_D f(x,y) \, dx \, dy.
\]

For the integral on the left hand side,

\[
f(x(u,v), y(u,v)) = f(u,u^2 + 2v) = \sqrt{u(u^2 + 2v)}.
\]

The Jacobian term:

\[
\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det \begin{pmatrix} 1 & 0 \\ 2u & 2 \end{pmatrix} = 2.
\]

So the integral on the left hand side is:

\[
\int_{u=0}^{u=1} \int_{v=0}^{v=u} \sqrt{u(u^2 + 2v)} \cdot 2 \, du \, dv.
\]

For the right hand side, we need \( D \). We parametrize each boundary segment of \( D^* \):

- For \( v = 0 \), we use \( t \mapsto (t,0) \), where \( 0 \leq t \leq 1 \). Applying our mapping to this gives \( T(t,0) = (t,t^2) \), which is the parabola \( y = x^2 \). Note that we’re interested in the portion between \((0,0)\) (corresponding to \( t = 0 \)) and \((1,1)\) (corresponding to \( t = 1 \)).
- For \( u = 1 \), we use \( t \mapsto (1,t) \), where \( 0 \leq t \leq 1 \). Applying \( T \), we get \( T(1,t) = (1,1+2t) \). This is the vertical line at \( x = 1 \), starting \( (t = 0) \) at \((1,1)\) and ending \( (t = 1) \) at \((1,3)\).
- For \( v = u \), we use \( t \mapsto (t,t) \), where \( 0 \leq t \leq 1 \). We have \( T(t,t) = (t,t^2 + 2t) \). So we want the segment of the parabola \( y = x^2 + 2x \) that lies between \((0,0)\) and \((1,3)\).

So \( D \) is the region in the first quadrant bounded below by \( y = x^2 \), above by \( y = x^2 + 2x \), and to the right by \( y = 1 \). Thus, the integral on the right hand side is:

\[
\int_{x=0}^{x=1} \int_{y=x^2}^{y=x^2+2x} \sqrt{xy} \, dx \, dy.
\]
4. (25 Points.) Compute the path integral of \( f(x, y) = 8x^3 \) over the graph \( y = 2x^3, \ 0 \leq x \leq 1 \).

We use the parametrization \( c(t) = (t, 2t^3), \ 0 \leq t \leq 1 \), for our path. Its velocity vector is:

\[
\vec{c}'(t) = \vec{i} + 6t^2 \vec{j}.
\]

So

\[
\|\vec{c}'(t)\| = \sqrt{1 + 36t^4}.
\]

The function translates to:

\[
f(x(t), y(t)) = f(t, 2t^3) = 8t^3.
\]

We integrate:

\[
\int_{c} f \ ds = \int_{t=0}^{t=1} f(x(t), y(t))\|\vec{c}'(t)\| \ dt
\]

\[
= \int_{t=0}^{t=1} 8t^3 \sqrt{1 + 36t^4} \ dt
\]

\[
= \left( \frac{1}{27} \frac{1}{2} \frac{2}{3} (1 + 36t^4)^{\frac{3}{2}} \right)_{t=0}^{t=1}
\]

\[
= \left( \frac{1}{27} (1 + 36t^4)^{\frac{3}{2}} \right)_{t=0}^{t=1}
\]

\[
= \frac{1}{27} 37^{\frac{3}{2}} - \frac{1}{27}.
\]