Math 20E Midterm I (ver. a)
Fall 2018

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/25</td>
</tr>
<tr>
<td>2</td>
<td>/26</td>
</tr>
<tr>
<td>3</td>
<td>/24</td>
</tr>
<tr>
<td>4</td>
<td>/25</td>
</tr>
<tr>
<td>Total</td>
<td>/100</td>
</tr>
</tbody>
</table>
1. (25 Points.) Evaluate the integral
\[
\int \int \int_W 1 \, dx \, dy \, dz,
\]
where \( W \) is the region determined by the conditions \( 1 \leq z \leq 5 \) and \( x^2 + y^2 + z^2 \leq 25 \).

\( W \) is the region of the sphere (of radius 5 about the origin) that lies above \( z = 1 \). We integrate using cylindrical coordinates.

\[
\int_\theta=0^{2\pi} \int_{r=0}^{\sqrt{24}} \int_{z=1}^{\sqrt{25-r^2}} 1 \, r \, dz \, dr \, d\theta.
\]

(The innermost integral goes from the plane to the sphere. The outer two integrals is over the disk of radius \( \sqrt{24} \) about the origin.) This is equal to

\[
\int_{\theta=0}^{2\pi} \int_{r=0}^{\sqrt{24}} \frac{1}{2} \sqrt{25-r^2} - r \, dr \, d\theta = \int_{\theta=0}^{2\pi} \left. \frac{8}{3} \right|_{r=0}^{\sqrt{24}} \, d\theta = \frac{8}{3} \cdot 2\pi = \frac{16\pi}{3}.
\]
2. (26 Points.)

(a) (13 Points.) Is the mapping \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by \( T(u, v) = (2u - 3v, 4u + v) \) onto? Justify your answer.

(b) (13 Points.) Let \( D^* = [0, 2] \times [0, 1] \) and define \( U : D^* \rightarrow \mathbb{R}^2 \) by \( U(u, v) = (u, -v^2 + 9v) \). Is \( U \) one-to-one? Justify your answer.

(a) This is a linear mapping corresponding to the linear mapping with determinant

\[
\det \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} = 2 + 12 = 14 \neq 0.
\]

Therefore it is onto (and one-to-one as well).

Without appealing to Linear Algebra: We need to show \( T(u, v) = (x, y) \) has a solution for all \((x, y) \in \mathbb{R}^2\). So we have to show that the system (with unknowns \( u \) and \( v \)):

\[
\begin{align*}
2u - 3v &= x \\
4u + v &= y
\end{align*}
\]

is solvable for all \( x \) and \( y \). We do this by explicitly finding the solutions.

The second equation tells us \( v = y - 4u \). Plugging this into the first equation gives \( 2u - 3(y - 4u) = x \), so that \( 14u - 3y = x \), giving us \( u = \frac{1}{14} x + \frac{3}{14} y \). Therefore \( v = y - \frac{2}{7} x - \frac{6}{7} y = -\frac{2}{7} x + \frac{1}{7} y \). And so we conclude

\[ T \left( \frac{1}{14} x + \frac{3}{14} y, -\frac{2}{7} x + \frac{1}{7} y \right) = (x, y). \]

This is valid for all choices of \( x \) and \( y \), telling us \( T \) is onto.

(b) There are many ways to do this one.

- Let’s start with \( U(u_1, v_1) = U(u_2, v_2) \). If this implies that \((u_1, v_1) = (u_2, v_2)\) then the mapping is one-to-one. We have:

\[ (u_1, -v_1^2 + 9v_1) = U(u_1, v_1) = U(u_2, v_2) = (u_2, -v_2^2 + 9v_2). \]

So we immediately conclude \( u_1 = u_2 \). We also have

\[ -v_1^2 + 9v_1 = -v_2^2 + 9v_2. \]

Throwing everything to the right hand side, we have

\[ 0 = v_1^2 - 9v_1 - v_2^2 + 9v_2 = v_1^2 - v_2^2 - 9(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2) - 9(v_1 - v_2) = (v_1 - v_2)(v_1 + v_2) - 9. \]

But \( v_1 + v_2 = 9 \) can never be equal to zero since \( D^* = [0, 2] \times [0, 1] \), which means both \( v_1 \) and \( v_2 \) must be less than \( 1 \). Therefore, \( v_1 - v_2 = 0 \), which means \( v_1 = v_2 \).

- Alternatively, solve for \( U(u, v) = (x, y) \) explicitly. You’ll get \( u = x \) and

\[ v = \frac{-9 \pm \sqrt{81 - 4y}}{2}. \]

The solution with the plus-sign in front of the root is not viable because:

\[ v = \frac{9 + \sqrt{81 - 4y}}{2} \geq \frac{9}{2} > 1, \]

which would be outside our legal range of \([0, 1]\) for \( v \). Therefore our equation can have at most one solution, making it one-to-one.

- Alternatively, you can say \(-v^2 + 9v\) is an increasing function (take its derivative) on \([0, 1]\) so it must be one-to-one there (officially, you’d quote the Mean Value Theorem).
3. (24 Points.)

- Let \( T(u, v) = (x(u, v), y(u, v)) \) be the mapping defined by \( T(u, v) = (u, u^2 + 2v) \).
- Let \( D^* \) be the triangle bounded by the three lines \( u = 1, v = 0, \) and \( v = u \).
- Let \( D = T(D^*) \) (the image of \( D^* \) under the mapping \( T \)).
- Let \( f(x, y) = \sqrt{xy} \) be a function defined on \( D \).

The Change of Variables/Coordinates Theorem tells us the integral of \( f \) over the region \( D \) is equal to an integral over the region \( D^* \).

(a) (12 Points.) Set up, **but do not compute** the integral on the left side of the equality.

(b) (12 Points.) Set up, **but do not compute** the integral on the right side of the equality.

Your integrals should have complete limits of integration; your integrands should be explicitly expressed in terms of the variables of integration.

**The Change of Variables Theorem:**

\[
\int \int_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv = \int \int_{D} f(x, y) \, dx \, dy.
\]

For the integral on the left hand side,

\[
f(x(u, v), y(u, v)) = f(u, u^2 + 2v) = \sqrt{u(u^2 + 2v)}.
\]

The Jacobian term:

\[
\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left| \begin{array}{cc} x_u & x_v \\ y_u & y_v \end{array} \right| = \left| \begin{array}{cc} 1 & 0 \\ 2u & 2 \end{array} \right| = 2.
\]

So the integral on the left hand side is:

\[
\int_{v=0}^{v=u} \int_{u=0}^{u=1} \sqrt{u(u^2 + 2v)} \cdot 2 \, du \, dv.
\]

For the right hand side, we need \( D \). We parametrize each boundary segment of \( D^* \):

- For \( v = 0 \), we use \( t \mapsto (t, 0) \), where \( 0 \leq t \leq 1 \). Applying our mapping to this gives \( T(t, 0) = (t, t^2) \), which is the parabola \( y = x^2 \). Note that we’re interested in the portion between \( (0, 0) \) (corresponding to \( t = 0 \)) and \( (1, 1) \) (corresponding to \( t = 1 \)).
- For \( u = 1 \), we use \( t \mapsto (1, t) \), where \( 0 \leq t \leq 1 \). Applying \( T \), we get \( T(1, t) = (1, 1 + 2t) \). This is the vertical line at \( x = 1 \), starting \( (t = 0) \) at \( (1, 1) \) and ending \( (t = 1) \) at \( (1, 3) \).
- For \( v = u \), we use \( t \mapsto (t, t) \), where \( 0 \leq t \leq 1 \). We have \( T(t, t) = (t, t^2 + 2t) \). So we want the segment of the parabola \( y = x^2 + 2x \) that lies between \( (0, 0) \) and \( (1, 3) \).

So \( D \) is the region in the first quadrant bounded below by \( y = x^2 \), above by \( y = x^2 + 2x \), and to the right by \( y = 1 \). Thus, the integral on the right hand side is:

\[
\int_{x=0}^{x=1} \int_{y=x^2}^{y=x^2+2x} \sqrt{xy} \, dx \, dy.
\]
4. (25 Points.) Compute the path integral of \( f(x,y) = 8x^3 \) over the graph \( y = 2x^3 \), \( 0 \leq x \leq 1 \).

We use the parametrization \( c(t) = (t, 2t^3) \), \( 0 \leq t \leq 1 \), for our path. Its velocity vector is:

\[
\vec{c}'(t) = \hat{i} + 6t^2 \hat{j}
\]

So

\[
\|\vec{c}'(t)\| = \sqrt{1 + 36t^4}.
\]

The function translates to:

\[ f(x(t), y(t)) = f(t, 2t^3) = 8t^3. \]

We integrate:

\[
\int_c f\,ds = \int_{t=0}^{t=1} f(x(t), y(t))\|\vec{c}'(t)\|\,dt
\]

\[
= \int_{t=0}^{t=1} 8t^3 \sqrt{1 + 36t^4} \,dt
\]

\[
= \left( \frac{1}{36} \frac{1}{2} \frac{2}{3} (1 + 36t^4)^{3/2} \right)\bigg|_{t=0}^{t=1}
\]

\[
= \left( \frac{1}{27} (1 + 36t^4)^{3/2} \right)\bigg|_{t=0}^{t=1}
\]

\[
= \frac{1}{27} 37^{3/2} - \frac{1}{27}.
\]