2.3.12

Follow the outline (p. 136) and using the following inequalities:

1) $\|Ax\| \le \|A\| . \|x\|$, $A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^{n\times 1}$ (see 2.1.25, p.

2) $\|x+y\| \le \|x\| + \|y\|$ (triangle inequality)

$\|A+B\| \le \|A\| + \|B\|$

2.3.13

Use 2.3.12 a):

$$\|\delta x\| \le \|A^{-1}\| \left(\|\delta b\| + \|\delta A\| \|\hat{x}\|\right) = \frac{\mathcal{K}(A)}{\|A\|}\left(\|\delta b\| + \|\delta A\| \left(\|x\| + \|\delta x\|\right)\right)$$

$$= \mathcal{K}(A) \cdot \left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|} \cdot \|x\| + \frac{\|\delta A\|}{\|A\|} \cdot \|\delta x\|\right)$$

Collecting $\|\delta x\|$ from RHS.

$$\|\delta x\| \underbrace{\left(1 - \mathcal{K}(A) \cdot \frac{\|\delta A\|}{\|A\|}\right)}_{>0} \le \mathcal{K}(A) \left(\frac{\|\delta b\|}{\|A\|} + \frac{\|\delta A\|}{\|A\|} \|x\|\right)$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \le \frac{\mathcal{K}(A) \cdot \left(\frac{\|\delta b\|}{\|A\|} \cdot \frac{1}{\|x\|} + \frac{\|\delta A\|}{\|A\|}\right)}{1 - \mathcal{K}(A) \cdot \frac{\|\delta A\|}{\|A\|}} \quad (1)$$

Since $Ax = b \Rightarrow \|b\| = \|Ax\| \le \|A\| . \|x\|$

$$\Rightarrow \frac{1}{\|x\|} \le \frac{\|A\|}{\|b\|} \quad (2)$$

Plug (2) in (1), we have the final result. $\square$

2.5.6

a) is equivalent to showing $(A+\delta A)\hat{x} = b$ 117)

$$LHS = \Big(A + \underbrace{\frac{1}{\|\hat{x}\|_2^2}(b - A\hat{x})\,\hat{x}^T}_{\delta A}\Big)\,\hat{x}$$

$$= A\hat{x} + \frac{1}{\|\hat{x}\|_2^2}(b - A\hat{x})\underbrace{(\hat{x}^T\hat{x})}_{\|\hat{x}\|_2^2}$$

$$= A\hat{x} + b - A\hat{x}$$

$$= b$$

$$= RHS \quad . \square$$

b) showing $\|\delta A\|_2 = \dfrac{\|\hat{r}\|_2}{\|\hat{x}\|_2}$ (✱)

⊕ By def $\|\delta A\|_2 = \sup\limits_{\substack{y \neq 0 \\ y \in \mathbb{R}^n}} \dfrac{\|(\delta A)\,y\|_2}{\|y\|_2}$,

Let $y = \hat{x}$ (assume $\hat{x} \neq 0$), $\|\delta A\|_2 \underset{(1)}{\geq} \dfrac{\|(\delta A)\,\hat{x}\|_2}{\|\hat{x}\|_2} = \dfrac{\|\hat{r}\|_2}{\|\hat{x}\|_2}$ (why?)

⊕ $\|\delta A\|_2 = \left\| \dfrac{1}{\|\hat{x}\|_2^2}(b - A\hat{x})\hat{x}^T \right\|_2 \underset{(2)}{\leq} \dfrac{1}{\|\hat{x}\|_2}\|b - A\hat{x}\|_2$. (why?).

From (1) and (2) $\Rightarrow$ (✱).

(✱) implies the other identity. $\square$

1.7.10

Similar to example 1.7.8 (p. 74)

1.7.18

Similar to 1.7.17 (p. 79) but need to justify more!