$\boxed{3.2.3}$ a) Need to show $(Q^{-1})(Q^{-1})^T = I$.

Since Q is orthogonal $Q^{-1} = Q^T$, $(Q^{-1})(Q^{-1})^T = Q^T (Q^T)^T = Q^T Q = I$

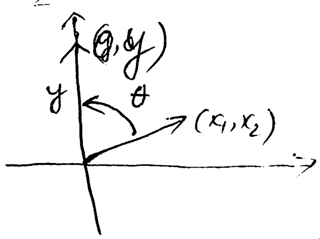
b) $(Q_1 Q_2)(Q_1 Q_2)^T = Q_1 \underbrace{Q_2 Q_2^T}_{I} Q_1^T = Q_1 Q_1^T = I$

$\boxed{3.2.8}$ (*) By defenition, $\|Q\|_2 = \max\limits_{x \neq 0} \dfrac{\|Qx\|_2}{\|x\|_2} \overset{\text{by 3.2.6}}{=} \max\limits_{x \neq 0} \dfrac{\|x\|_2}{\|x\|_2} = 1$

(*) Q is orthogonal $\Rightarrow$ Q^{-1} is orthogonal. Similarly $\|Q^{-1}\|_2 = 1$

(*) $K_2(Q) = \|Q\|_2 \cdot \|Q^{-1}\|_2 = 1 \cdot 1 = 1$

$\boxed{3.2.12}$ a)



$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is rotated through an angle $\theta = tg^{-1}\left(\frac{x_1}{x_2}\right)$

b) $Q^T x = \begin{bmatrix} y \\ 0 \end{bmatrix} \Rightarrow \|Q^T x\|_2 = \left\|\begin{bmatrix} y \\ 0 \end{bmatrix}\right\|_2 \Rightarrow \|x\|_2 = \sqrt{y^2}$ (since Q^T is orthogonal)

$\Rightarrow |y| = \sqrt{x_1^2 + x_2^2}$ (The book misses the absolute signs. y can be negative).

$\boxed{3.2.16}$

(*) $$Q Q^T = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & -s & \\ & & & \ddots & & \\ & & s & & c & \\ & & & & & \ddots \ 1 \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & & s & \\ & & & \ddots & & \\ & & -s & & c & \\ & & & & & \ddots \ 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c^2+s^2 & & cs-sc & \\ & & & \ddots & & \\ & & sc-cs & & c^2+s^2 & \\ & & & & & \ddots \ 1 \end{bmatrix} = I$$

(*) Prove by induction on the matrix size n.

- Base case $n = 2$ $\begin{vmatrix} c & -s \\ s & c \end{vmatrix} = c^2 + s^2 = 1$
- To go from $n = k+1$ to $n = k$ use Determinant Expansion by Minors for a row (or colum) containing only 0 and 1.

3.2.14

$$\underbrace{\begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & -s & & \\ & & & \ddots & & & \\ & & s & & c & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ cx_i - sx_j \\ \vdots \\ sx_i + cx_j \\ \vdots \\ x_n \end{bmatrix}}_{Qx}, \quad \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} = \begin{bmatrix} cx_i - sx_j \\ sx_i + cx_j \end{bmatrix}$$

3.2.27

a) $Qu = (I - 2uu^T)u = u - 2u(u^Tu) = u - 2u\underbrace{\|u\|_2^2}_{1} = u - 2u = -u$

b) $Qv = (I - 2uu^T)v = v - 2u\underbrace{(u^Tv)}_{0} = v - 0 = v$

c) $Q^T = (I - 2uu^T)^T = I - 2(u^T)^Tu^T = I - 2uu^T = Q$

d) $Q^T \cdot Q = (I - 2uu^T)(I - 2uu^T) = I - 2uu^T - 2uu^T + 4u\underbrace{u^Tu}_{1}u^T$

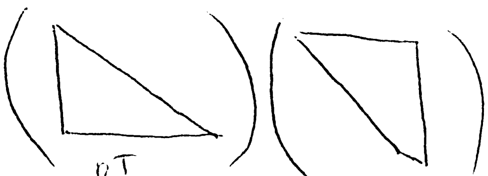
$= I - 4uu^T + 4uu^T = I$

$\Rightarrow Q^T = Q^{-1}$

e) $Q^{-1} \underset{d)}{=} Q^T \underset{c)}{=} Q \Rightarrow Q^{-1} = Q$

3.2.47 Let $A = B^TB$ and $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $B^T = (b^T_{ij})_{n\times n}$

Since B is orthogonal $\Rightarrow A = I$

$B^TB =$ 

B^T B

- Look at $(1,1)$

$1 = a_{11} = b^T_{11} \cdot b_{11} = b_{11}^2 = 1 \Rightarrow b_{11} = \pm 1 \neq 0$

- For $j \neq 1$,

$0 = a_{1j} = b^T_{11} \cdot b_{1j} = b_{11} \cdot b_{1j}$

$\Rightarrow b_{1j} = 0, \ \forall j \neq 1$

- Look at $(2,2)$

$1 = a_{22} = b^T_{21} b_{12} + b^T_{22} b_{22} = \underbrace{b_{12} \cdot b_{12}}_{0} + b_{22}^2 \Rightarrow b_{22} = \pm 1 \neq 0$

- For $j \neq 2$:

$0 = a_{2j} = b^T_{21} b_{1j} + b^T_{22} b_{2j} = \underbrace{b_{12} \cdot b_{1j}}_{0} + \underbrace{b_{22}}_{\neq 0} \cdot b_{2j}$

$\Rightarrow b_{2j} = 0$

Continue this procedure we have $b_{ij} = 0$ for $i \neq j$

$b_{ii} = \pm 1$

So B is a diagonal matrix with diagonal entries are ± 1.

b) $Q_1 R_1 = Q_2 R_2$

Multiply both sides with Q_1^{-1} (from the left) and R_2^{-1} (from the right) (why Q_1^{-1}, R_2^{-1} exist?).

$$Q_1^{-1}(Q_1 R_1) R_2^{-1} = Q_1^{-1}(Q_2 R_2) R_2^{-1}$$

$$\Leftrightarrow \quad R_1 R_2^{-1} = Q_1^{-1} Q_2 \quad (*)$$

From previous homework, $\cancel{R_1}\,\cancel{R_2}$ R_2^{-1} is upper ∇; R_1 ——— ∇ $\Big\} \Rightarrow R_1 R_2^{-1}$ is ∇

Q_i is orthogonal $\Rightarrow Q_1^{-1}$ is orthogonal; Q_2 ——— $\Big\} \Rightarrow Q_1^{-1} Q_2$ is orthogonal

Use a) and $(*)$ $\Rightarrow \quad R_1 R_2^{-1} = Q_1^{-1} Q_2 = D$

where D is diagonal ~~with~~ whole entries on the diagonal are ± 1

$$\Rightarrow \quad R_1 = D R_2 \; ; \; Q_2 = Q_1 D$$