The following integrals and integral rules might be useful for the test:

- **Chain Rule:** If $f$ is a function of $u$, and $u$ is a function of $x$, then
  \[
  \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}
  \]

- **Product Rule:** If $u$ and $v$ are functions of $x$, then
  \[
  \frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}
  \]

- **Useful integrals:**
  \[
  \int (ax + b)^n \, dx = \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n + 1} + C,
  \]
  \[
  \int e^{ax+b} \, dx = \frac{1}{a} \cdot e^{ax+b} + C,
  \]
  \[
  \int \sin(ax + b) \, dx = -\frac{1}{a} \cdot \cos(ax + b) + C,
  \]
  \[
  \int \cos(ax + b) \, dx = \frac{1}{a} \cdot \sin(ax + b) + C,
  \]
  \[
  \int \frac{1}{ax + b} \, dx = \frac{1}{a} \cdot \ln |ax + b| + C,
  \]
  \[
  \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C,
  \]
  \[
  \int \frac{1}{1 + x^2} \, dx = \tan^{-1} x + C,
  \]
  \[
  \int \frac{1}{\sqrt{1 + x^2}} \, dx = \sinh^{-1} x + C,
  \]

The first order Ordinary Differential Equations are of the form $\frac{dy}{dt} = f(y, t)$. In particular,

- if $\frac{dy}{dt} = g(y)h(t)$, it is called a **separable** equation. To solve it, separate the two variables, and take integrals on both sides:
  \[
  \frac{1}{g(y)} \, dy = h(t) \, dt \rightarrow \int \frac{1}{g(y)} \, dy = \int h(t) \, dt + C.
  \]

- if $\frac{dy}{dt} + p(t)y = g(t)$, it is called a **linear** equation. To solve it, multiply the two sides by an integrating factor $\mu(t)$, which can be found by $\mu = e^{\int p(t) \, dt}$, then the equation becomes:
  \[
  \mu y' + \mu p(t)y = \mu g(t) \rightarrow (\mu y)' = \int \mu g \, dt + C \rightarrow y = \frac{1}{\mu} \left( \int \mu g \, dt + C \right)
  \]

- if $M(x, y) + N(x, y)y' = 0$, and $M_y(x, y) = N_x(x, y)$, it is called an **exact** equation. To solve it, set up a system of partial differential equations:
  \[
  \phi_x(x, y) = M(x, y), \quad \phi_y(x, y) = N(x, y)
  \]
  and solve for the **potential** function $\phi$. Then the solution is $\phi(x, y) = C$. 
Existence and uniqueness of solutions.

- Given an initial value problem
  \[y' + p(t)y = g(t), \quad y(t_0) = y_0,\]
  if the functions \(p(t)\) and \(g(t)\) are continuous on some interval \((a, b)\), and \(t_0 \in (a, b)\), then this initial value problem has a unique solution.

- Given an initial value problem
  \[y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0\]
  if the functions \(p(t), q(t)\) and \(g(t)\) are continuous on some interval \((a, b)\), and \(t_0 \in (a, b)\), then this initial value problem has a unique solution.

Autonomous Equations and stability analysis on their equilibrium solutions, memorize them.

The second order linear, homogeneous, constant-coefficient ODEs are of form
\[ay'' + by' + cy = 0,\]
which can be solved by considering the characteristic equation
\[ar^2 + br + c = 0.\]

- If the characteristic equation has two distinct real roots \(r_1, r_2\), then the ODE has two independent solutions \(y_1 = e^{r_1t}, y_2 = e^{r_2t}\);

- If the characteristic equation has two complex roots \(r_{1,2} = \lambda \pm i\mu\), then the ODE has two independent solutions \(y_1 = e^{\lambda t}\cos(\mu t), y_2 = e^{\lambda t}\sin(\mu t)\);

- If the characteristic equation has two equal roots \(r_1 = r_2 = -b/(2a)\), then the ODE has two independent solutions \(y_1 = e^{-rt}, y_2 = te^{-rt}\);

and the general solution is of the form \(y = c_1y_1 + c_2y_2\).

Two solutions \(y_1, y_2\) of the equation \(y'' + p(t)y' + q(t)y = 0\) are independent if and only if the Wronskian
\[W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2\]
is nonzero.
**Reduction of Order**: Suppose we know one nonzero solution $y_1(t)$ of the second order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$. To find another solution $y_2(t)$ which is independent of $y_1(t)$, we have a few steps

1. Assume $y_2 = v(t)y_1(t)$;
2. Insert $y_2 = v(t)y_1(t)$ into the equation $y'' + p(t)y' + q(t)y = 0$ to have a simplified equation
   
   $y_1v'' + (2y_1' + py_1)v' = 0$;
3. Solve the $v$-equation in step 2 by changing it into a system of first order ODEs
   
   $y_1w' + (2y_1' + py_1)w = 0, \quad v' = w$
4. After finding the $v$, plug it back into $y_2 = v(t)y_1(t)$, and choose a simple $y_2$ which is independent of $y_1$. Verify the independence of $y_1, y_2$ by calculating the Wronskian.