Useful integrals or formulae:
\[
\begin{align*}
\int (ax + b)^n \, dx &= \frac{1}{a} \cdot \frac{(ax + b)^{n+1}}{n+1} + C, \\
\int e^{ax+b} \, dx &= \frac{1}{a} \cdot e^{ax+b} + C, \\
\int \sin(ax + b) \, dx &= -\frac{1}{a} \cdot \cos(ax + b) + C, \\
\int \cos(ax + b) \, dx &= \frac{1}{a} \cdot \sin(ax + b) + C, \\
\int \frac{1}{ax + b} \, dx &= \frac{1}{a} \cdot \ln |ax + b| + C, \\
\int \frac{1}{\sqrt{1-x^2}} \, dx &= \sin^{-1} x + C, \\
\int \frac{1}{1+x^2} \, dx &= \tan^{-1} x + C, \\
\int \frac{1}{\sqrt{1+x^2}} \, dx &= \sinh^{-1} x + C,
\end{align*}
\]

Chain Rule: \[ \frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}, \]

Product Rule: \[ \frac{d(u \cdot v)}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}, \]

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] together with \( ad - bc \neq 0 \) implies \( A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

The first order Ordinary Differential Equations are of the form \( \frac{dy}{dt} = f(y, t) \). In particular,

- if \( \frac{dy}{dt} = g(y)h(t) \), it is called a **separable** equation. To solve it, separate the two variables, and take integrals on both sides:
  \[ \frac{1}{g(y)} \, dy = h(t) \, dt \rightarrow \int \frac{1}{g(y)} \, dy = \int h(t) \, dt + C. \]

- if \( \frac{dy}{dt} + p(t)y = g(t) \), it is called a **linear** equation. To solve it, multiply both sides by an integrating factor \( \mu(t) \), which can be found by \( \mu = e^{\int p(t) \, dt} \), then the equation becomes:
  \[ \mu y' + \mu p(t)y = \mu g(t) \rightarrow (\mu y)' = \mu g(t) \rightarrow \mu y = \int \mu g(t) \, dt + C \rightarrow y = \frac{1}{\mu} \left( \int \mu g(t) \, dt + C \right) \]

- if \( M(x, y) + N(x, y) y' = 0 \), and \( M_y(x, y) = N_x(x, y) \), it is called an **exact** equation. To solve it, set up a system of partial differential equations:
  \[ \phi_x(x, y) = M(x, y), \quad \phi_y(x, y) = N(x, y) \]
and solve for the **potential** function \( \phi \). Then the solution is \( \phi(x, y) = C \).
The second order linear, homogeneous, constant-coefficient ODEs are of form

\[ ay'' + by' + cy = 0, \]

which can be solved by considering the characteristic equation

\[ ar^2 + br + c = 0. \]

- If the characteristic equation has two distinct real roots \( r_1, r_2 \), then the ODE has two independent solutions \( y_1 = e^{r_1 t}, y_2 = e^{r_2 t} \);
- If the characteristic equation has two complex roots \( r_{1,2} = \lambda \pm i\mu \), then the ODE has two independent solutions \( y_1 = e^{\lambda t} \cos(\mu t), y_2 = e^{\lambda t} \sin(\mu t) \);
- If the characteristic equation has two equal roots \( r_1 = r_2 = -b/(2a) \), then the ODE has two independent solutions \( y_1 = e^{r_1 t}, y_2 = te^{r_1 t} \);

and the general solution is of the form \( y = c_1 y_1 + c_2 y_2 \).

The second order linear, nonhomogeneous, ODEs are of form

\[ y'' + p(t)y' + q(t)y = g(t). \]

They can be solved using:

- **Undetermined Coefficient Method:** Solve the corresponding homogeneous equation for two independent solutions \( y_1(t) \) and \( y_2(t) \). Guess a particular solution \( y_p(t) \) according to \( g(t) \). Insert \( y_p(t) \) back into the nonhomogeneous ODE, and get linear equations on the undetermined coefficients. Solve for these coefficients. And the general solution is

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t). \]

- **Variation of Parameters Method:** A particular solution is given by the following formula:

\[ y_p(t) = -y_1 \int \frac{y_2(t)g(t)}{W(y_1, y_2)} \, dt + y_2 \int \frac{y_1(t)g(t)}{W(y_1, y_2)} \, dt, \]

where \( W(y_1, y_2) = y_1 y_2' - y_2 y_1' \) is the Wronskian, and the general solution is given by

\[ y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t). \]
For a 2-by-2 system of ODEs
\[ x' = Ax, \]
where \( A \) is an 2-by-2 matrix, and \( x \) is a 2-by-1 vector, there are 2 linearly independent solutions \( u_1, u_2 \). If their Wonskian is nonzero, that is,
\[ W(u_1, u_2) = \det(u_1, u_2) \neq 0, \]
then these solutions form a fundamental set of solutions, that is, \( u_1, u_2 \) are linearly independent. Then the general solution is
\[ x(t) = c_1 u_1 + c_2 u_2. \]
For initial value problems
\[ x' = Ax, \quad \text{with} \quad x(t = 0) = x_0, \]
you need to determine \( c_1 \) and \( c_2 \) by \( c_1 u_1(0) + c_2 u_2(0) = x_0 \), i.e., you need to solve the linear system of equations
\[ (u_1(0), u_2(0)) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

To solve for a 2-by-2 homogeneous, linear constant coefficient system of ODEs
\[ x' = Ax, \]
first solve for the characteristic equation \( \det(A - \lambda I) = 0 \) to get two eigenvalues.

- if the two eigenvalues \( \lambda_1, \lambda_2 \) are real and distinct, then you get a stable node (if \( \lambda_1 < 0 \) and \( \lambda_2 < 0 \)), or an unstable node (if \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \), or a saddle point (if \( \lambda_1 \lambda_2 < 0 \)). The general solution is given by
\[ x = c_1 u_1 e^{\lambda_1 t} + c_2 u_2 e^{\lambda_2 t}, \]
where \( u_1 \) and \( u_2 \) are two linearly independent eigenvectors of \( A \) corresponding to \( \lambda_1 \) and \( \lambda_2 \), i.e., \((A - \lambda_1 I) \cdot u_1 = 0 \) and \((A - \lambda_2 I) \cdot u_2 = 0 \).

- if the two eigenvalues are complex, that is \( \lambda = \alpha \pm i\beta \), then you get a stable spiral (if \( \alpha < 0 \)) or an unstable spiral (if \( \alpha > 0 \)). To find the solution, solve for the eigenvector \( u_1 = v_1 + iv_2 \) corresponding to \( \lambda_1 = \alpha + i\beta \). Then the second eigenvector corresponding to \( \lambda_2 = \alpha - i\beta \) is \( u_2 = v_1 - iv_2 \). And the general solution is given by
\[ x = c_1 (v_1 \cos \beta t - v_2 \sin \beta t) e^{\alpha t} + c_2 (v_1 \sin \beta t + v_2 \cos \beta t) e^{\alpha t}. \]

- if the two eigenvalues are repeated, i.e., \( \lambda_1 = \lambda_2 = \lambda \), there are two possibilities:
- **A stable (if \( \lambda < 0 \)) or **unstable (if \( \lambda > 0 \)) node**: if you can obtain two linearly independent eigenvectors, \( u_1 \) and \( u_2 \), then the general solution is given by
  \[
  x = c_1 u_1 e^{\lambda t} + c_2 u_2 e^{\lambda t}.
  \]

- **A stable (if \( \lambda < 0 \)) or **unstable (if \( \lambda > 0 \)) improper node**: if you can only obtain one linearly independent eigenvector, \( u \), then you need to find a generalized eigenvector \( v \), such that \((A - \lambda I)v = u\). And the general solution is
  \[
  x = c_1 u e^{\lambda t} + c_2 (v + ut) e^{\lambda t}.
  \]

To solve for 2-by-2 nonhomogeneous, linear constant coefficient systems of ODEs, i.e.,
\[
x' = Ax + g(t),
\]
there are four methods, but we use only two of them:

- **Diagonalization Method**: If there are two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then you can find two linearly independent eigenvectors \( u_1 \) and \( u_2 \).
  - Let \( P = [u_1, u_2] \), find \( P^{-1} \).
  - Let \( D = P^{-1} \cdot A \cdot P \), in fact, \( D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \).
  - Let \( x = P \cdot y \), then we have a new system of ODEs on \( y \)
    \[
    y' = Dy + h(t), \quad \text{with} \quad h(t) = P^{-1} \cdot g(t)
    \]
  - Now solve for the decoupled system \( y_1' = \lambda_1 y_1 + h_1(t) \) and \( y_2' = \lambda_2 y_2 + h_2(t) \) to get \( y \).
  - To solve for \( y_1 \) and \( y_2 \), you can use the formula for first order linear ODEs.
  - Compute \( x \) using \( x = P \cdot y \). Don’t forget to include the two constants \( c_1 \) and \( c_2 \).

- **Undetermined Coefficient Method**: This method is similar to the undetermined coefficient method for 2nd order ODEs.
  - You solve for the homogeneous system first, obtain the solution \( x_h \),
  - then you take a guess on the particular solution \( x_p \). The guess \( x_p \) may be a linear combination (that is, an addition) of coefficient vectors times a function such as \( t, e^{rt} \) or \( \sin \beta t, \cos \beta t \), etc.
  - Insert \( x_p \) back into the equation \( x' = Ax + g(t) \), and get a group of linear equations.
  - Solve for these linear equations, get the coefficient vectors, and put them back to the \( x_p \).
  - The general solution is \( x = x_h + x_p \).

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### Table 1: Elementary Laplace Transforms

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s) = \mathcal{L}{f(t)}$</th>
<th>$F(s) = \mathcal{L}^{-1}{F(s)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $1$</td>
<td>$\frac{1}{s}$, $s &gt; 0$</td>
<td>$1$</td>
</tr>
<tr>
<td>2. $e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
<td>$e^{at}$</td>
</tr>
<tr>
<td>3. $t^n$, $n$ positive integer</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
<td>$t^n$</td>
</tr>
<tr>
<td>4. $t^p$, $p &gt; -1$</td>
<td>$\frac{\Gamma(p + 1)}{s^{p+1}}$, $s &gt; 0$</td>
<td>$t^p$</td>
</tr>
<tr>
<td>5. $\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$, $s &gt; 0$</td>
<td>$\sin at$</td>
</tr>
<tr>
<td>6. $\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
<td>$\cos at$</td>
</tr>
<tr>
<td>7. $\sinh at$</td>
<td>$\frac{a}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>8. $\cosh at$</td>
<td>$\frac{s}{s^2 - a^2}$, $s &gt;</td>
<td>a</td>
</tr>
<tr>
<td>9. $e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$, $s &gt; a$</td>
<td>$e^{at} \sin bt$</td>
</tr>
<tr>
<td>10. $e^{at} \cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$, $s &gt; a$</td>
<td>$e^{at} \cos bt$</td>
</tr>
<tr>
<td>11. $t^ne^{at}$, $n$ positive integer</td>
<td>$\frac{n!}{(s-a)^{n+1}}$, $s &gt; a$</td>
<td>$t^ne^{at}$</td>
</tr>
<tr>
<td>12. $u_c(t)$</td>
<td>$e^{-cs}$, $s &gt; 0$</td>
<td>$u_c(t)$</td>
</tr>
<tr>
<td>13. $u_c(t)f(t-c)$</td>
<td>$e^{-cs}F(s)$</td>
<td>$u_c(t)f(t-c)$</td>
</tr>
<tr>
<td>14. $e^{ct}f(t)$</td>
<td>$F(s-c)$</td>
<td>$e^{ct}f(t)$</td>
</tr>
<tr>
<td>15. $f(ct)$</td>
<td>$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c &gt; 0$</td>
<td>$f(ct)$</td>
</tr>
<tr>
<td>16. $\delta(t-c)$</td>
<td>$e^{-cs}$</td>
<td>$\delta(t-c)$</td>
</tr>
<tr>
<td>17. $f^{(n)}(t)$</td>
<td>$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$</td>
<td>$f^{(n)}(t)$</td>
</tr>
</tbody>
</table>