SOLUTION SKETCHES AND/OR ANSWERS TO SOME HOMEWORK PROBLEMS

The numbers refer to the 7th edition. Compare with the homework assignment if you are using the fifth edition. The style is somewhat terse, and ideally there should be more details. But I hope this will help anway.

Chapter 13, # 24 (a) Any nonzero function f for which there exists an $a \in \mathbf{R}$ for which $f(a) = 0$ is a zero divisor: indeed, just take the function $g(x)$ such that $g(a) = 1$ and $g(x) = 0$ if $x \neq a$. Then also g is a nonzero function and $f(x)g(x) = 0$ for all $x \in \mathbf{R}$.

For (b), there is no nilpotent element in R. If $f(a) \neq 0$ for some $a \in \mathbf{R}$, then also $f^n(a) = f(a)^n \neq 0$. Hence if $f \neq 0$, then so is f^n.

For (c), we have for any nonzero function f either that there exists an $a \in \mathbf{R}$ for which $f(a) = 0$ (in which case f is a zero divisor), or $f(a) \neq 0$ for all $a \in \mathbf{R}$. In the second case, we can define a function $g(x)$ by $g(a) = f(a)^{-1}$ for all $a \in \mathbf{R}$. It then follows that $f(a)g(a) = 1$ for all $a \in \mathbf{R}$, i.e. $f(x)g(x) = 1$, the constant function which is the multiplicative identity for the functions on $\mathbf{R}$.

Chapter 14, # 31 Let J be an ideal bigger than A. Then it must contain a function f for which $f(0) \neq 0$. Let $g(x)$ be a function which is qual to 1 if $x \neq 0$ and $g(0) = 0$. Then the function $h(x) = f(x)^2 + g(x)$ is positive for all values of x, and it is in J. But any ideal containing an invertible element h (i.e. a unit) must contain every element of the ring (proof: Let $b \in R$; then we also have $b = (bh^{-1})h \in J$).

Chapter 14, # 34 Let J be the ideal in $\mathbf{Z}[x]$ consisting of all polynomials with even constant term. (Check that this is indeed an ideal!). This contains the ideal I as for any $f \in I$ we have $0 = f(0) =$ constant term of f, and 0 is an even number. But J also contains the constant function 2, which is not in I. On the other hand, it does not contain the constant function 1, hence J is not equal to the whole ring $\mathbf{Z}[x]$.

Chapter 16, # 22 If $f - g$ is a nonzero polynomial, there would only exist finitely many elements a for which $(f - g)(a) = f(a) - g(a) = 0$ (namely at most as many as the degree of $f - g$). But then $f(a)$ could not be equal to $g(a)$ for infinitely many elements $a \in F$.

Show that there exist a, b in F such that $x^2 + x + 1$ divides $x^{43} + ax + b$. We know by long division that $x^{43} = q(x)(x^2 + x + 1) + r(x)$ with degree of $r(x) \leq 1$ or $r(x) = 0$. This means we can write $r(x) = cx + d$. But then $x^2 + x + 1$ divides $x^{43} - cx - d$, i.e. we can set $a = -c$ and $b = -d$.