1. Eighth assignment, second problem: Assume \( \alpha = [a_0, a_1, \ldots, a_k] \). Then also \( \alpha_1 = [a_1, a_2, \ldots, a_k, a_0] \) and all the other \( \alpha_j \)'s are purely periodic. By the information given in the assignment, this means that \(-1 < \alpha_j < 0\) for all \( j \geq 0 \). Now recall that

\[
\frac{1}{\alpha_{j+1}} = \alpha_j \iff \alpha_j = a_j + \frac{1}{\alpha_{j+1}} \iff \bar{\alpha}_j = a_j + \frac{1}{\alpha_{j+1}}.
\]

By periodicity, we have \( \alpha = \alpha_0 = \alpha_{k+1} \). Hence, if we set \( \beta = -\frac{1}{\bar{\alpha}} \), we have

\[
\beta = -\frac{1}{\bar{\alpha}_{k+1}} = a_k - \bar{\alpha}_k.
\]

As we have \( 0 < -\bar{\alpha}_k < 1 \), it follows that \( b_0 = a_k \) is the integer part of \( \beta \) and \( \beta_1 = -\frac{1}{\alpha_k} \). As \( \alpha_k = [a_k, a_0, \ldots, a_{k-1}] \) is again purely periodic, we show as before that the integer part \( b_1 \) of \( \beta_1 \) is equal to \( a_{k-1} \). Continuing this way, we get the claim.

2. Extra credit problem: At hindsight, I should have given you an additional hint which would have made it much easier to get to some explicit statements. In any case, if you managed to guess something like the statements below by try and error under these conditions, this would already be quite a good achievement. The following is a special case of Theorem 9.14 which is very useful in this context:

(*) If \( \beta \) is a quadratic irrational, then \( \beta \) and \( -1/\beta \) have the same periodic part (or, in the statement of Theorem 9.14, they are equivalent).

Now let \( \alpha \) be a quadratic rational. Then it is periodic, i.e. we can write it as \( \alpha = [a_0, a_1, \ldots, a_m, \beta] \) where \( \beta \) is purely periodic. The same formula holds if we put \( \tilde{\alpha} \) over both \( \alpha \) and \( \beta \). Now we know by Problem 2 above that \( -1/\beta \) has the reverse periodic part of \( \beta \). By statement (*) we also have that \( \beta \) has the same periodic part as \( -1/\beta \). Plugging this into the continued fraction part of \( \bar{\alpha} \), we see that also \( \bar{\alpha} \) has the same periodic part as \( -1/\beta \). Hence we get:

Result: If \( \alpha \) is a quadratic irrational, then the periodic part of \( \bar{\alpha} \) is the reverse of the periodic part of \( \alpha \). For an example, see Problem 3 in page 255 in the book: \((4 + \sqrt{10})/3 = [2, 2, 1, 1]\), while \((4 - \sqrt{10})/3 = [0, 3, 1, 1, 2]\).

3. First problem of seventh assignment: Observe that the numerators of the convergents of the continued fraction expansion are given by \( q_0 = 1, q_1 = 2 \) and \( q_k = 2k^2 q_{k-1} + q_{k-2} \). One deduces from this by induction that \( q_k \leq 2^{k-1} q^{k+1+2^k} + \ldots + k^1 \), which is the hint. One can also show by induction that \( k + 2! + 3! + \ldots + k! \leq 2k! \). Hence we have \( q_k \leq 2^{2k^2} \) and \( q_{k+1} \geq 2^{k^2} q_k \). It follows that

\[
q_{k+1} \geq \frac{(2k^2)^{k+1}}{2^{k+1}} q_k \geq 2^{k+1}(k+1)/2.
\]

If \( \alpha = [0, 2^1, 2^2, 2^3, \ldots] \) and \( C_k = \frac{p_k}{q_k} \) is its \( k \)-th convergent, we have

\[
|\alpha \frac{p_k}{q_k} | < |C_{k+1} - C_k| = \frac{1}{q_k q_{k+1}} < \frac{1}{2^{2(k+1)/2}}.
\]
Now if $\alpha$ was algebraic of degree $n$, we would have, by Liouville’s theorem, that

$$\frac{1}{q_k^{2+(k+1)/2}} > \left| \alpha - \frac{p_k}{q_k} \right| > \frac{M}{q_k^n}$$

for some fixed constant $M$ and for all $q_k$. But then we would have, after multiplying the expression above by $q_k^{2+(k+1)/2}$ that

$$1 > Mq_k^{2-n+(k+1)/2}$$

for all $k$. But this is not possible as $q_k \to \infty$ and $2 - n + (k + 1)/2 \to \infty$ for $k \to \infty$. 

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