FIFTH HOMEWORK ASSIGNMENT

You can use earlier statements of the assignment for the proof of later ones regardless whether you could prove them or not. If you get stuck, ask Joel or me for hints.

1. (a) Find all solutions of $x^2 - 2y^2 = 1$.

   (b) Find alll solutions of $x^2 - 2y^2 = 49$. Also show that in (20) of Theorem 8.9 in the book, one should have a $\leq$ sign also for the first inequality.

2. (a) Suppose that the fundamental solution of $x^2 - dy^2 = 1$ is given by $\delta = x_1 + \sqrt{d}y_1$. Show that the equation $x^2 - dy^2 = -1$ has a fundamental solution γ if and only if $\sqrt{(x_1 - 1)/2}$ and $y_1/2$ are integers and the first divides the second.

   (b) Calculate all solutions of $x^2 - 13y^2 = -1$ and $x^2 - 21y^2 = -1$, or give a reason why there are none. Hint: The fundamental solutions for $x^2 - 13y^2 = 1$ and $x^2 - 21y^2 = 1$ are given by $649 + 180\sqrt{13}$ and by $55 + 12\sqrt{21}$. (See also the remark about Section 8.2 on page 216).

3. Calculate all units in the ring of algebraic integers for $\mathbf{Q}(\sqrt{13})$ (Hint: By Theorems 8.12 and 8.11, this boils down to solving certain Pell's equations, see also Theorem 8.8. Finding a minimal solution now is much easier than in 2(b)).

4. (Extra credit - can be turned in later): Our analysis of algebraic integers, Theorem 8.11, also applies for $d < 0$. Find all units in the ring $\mathbf{Z}(F)$ of algebraic integers for $F = \mathbf{Q}(\sqrt{d})$ with $d < 0$. Hint: Use the norm $N(a + b\sqrt{d}) = a^2 - db^2$ also for $d < 0$. There are not many cases where we have more units than ± 1.