SEVENTH HOMEWORK ASSIGNMENT

1. Show that the number given by the continued fraction expansion \([0, 2^{11}, 2^{22}, 2^{33}, \ldots]\) is transcendental. (Hint: Show first that \(q_k \leq 2^{k-1}a_1a_2 \ldots a_k\) for \(k \geq 1\).)

2. (a) Calculate the continued fraction expansion for \(\sqrt{21}\) and \((2 + \sqrt{7})/4\) (It is enough to calculate \(a_i, i \leq 2\) for the second number). Read the beginning of Problem 3 first.
   (b) Evaluate the following continued fractions: \([3, 2, 1, 3, 2, 1, \ldots]\) and \([1, 2, 3, 2, 3, 2, 3, \ldots]\).

3. We have shown that if \(\alpha = \frac{A_0 + \sqrt{d}}{B_0}\) is a quadratic algebraic number, the elements \(\alpha_k\), which are defined inductively by \(\alpha_0 = \alpha, a_k = [\alpha_k]\) (integer part), and \(a_{k+1} = 1/(\alpha_k - a_k)\) can be written as
   \[\alpha_k = \frac{A_k + \sqrt{d}}{B_k},\]
   where \(A_{k+1} = a_kB_k - A_k\) and \(B_{k+1} = (d - A_{k+1}^2)/B_k\). Also recall that the convergents \(C_k = p_k/q_k\) are defined by \(p_0 = a_0, p_1 = a_0a_1 + 1, q_0 = 1\) and \(q_1 = a_1\), and by the recursion relations
   \[p_k = a_k p_{k-1} + p_{k-2}, \quad q_k = a_k q_{k-1} + q_{k-2}.\]
   (a) Show for the special case \(\alpha = \sqrt{d}\) that
   \[p_k^2 - dq_k^2 = (-1)^{k+1}B_{k+1}, \quad p_kp_{k-1} - dq_kq_{k-1} = (-1)^{k+1}A_{k+1}.\]
   (b) Assume that \(\sqrt{d}\) has the periodic continued fraction expansion \([a_0; a_1, a_2, \ldots, a_m]\) (i.e. \(a_{k+m} = a_k\) for \(k > 0\); proof of this on Monday). Show that \(p_{k+m}^2 - dq_{k+m}^2 = (-1)^m(p_k^2 - dq_k^2)\) for all \(k > 0\).
   (c) Prove that there exists a solution of Pell’s equation \(x^2 - dy^2 = (-1)^m\) of the form \(x = p_{m-1}, y = q_{m-1}\). (Hint: Periodicity only helps to compare \(\alpha_1\) with \(\alpha_{m+1}\); this should still help to figure out \(B_m\)).
   (d) Prove that \(x^2 - dy^2 = -1\) has a solution if and only if the period \(m\) for the continued fraction expansion for \(\sqrt{d}\) is odd. What about \(x^2 - 21y^2 = -1\)?