SEVENTH HOMEWORK ASSIGNMENT

1. Show that the number given by the continued fraction expansion $[0, 2^{1!}, 2^{2!}, 2^{3!}, \ ... \ ]$ is transcendental. (Hint: Show first that $q_k \leq 2^{k-1} a_1 a_2 \ ... \ a_k$ for $k \geq 1$.)

2. (a) Calculate the continued fraction expansion for $\sqrt{21}$ and $(2+\sqrt{7})/4$ (It is enough to calculate a_i, $i \leq 2$ for the second number). Read the beginning of Problem 3 first.
(b) Evaluate the following continued fractions: $[3, 2, 1, 3, 2, 1, \ ...]$ and $[1, 2, 2, 3, 2, 3, \ ....]$

3. We have shown that if $\alpha = \frac{A_0+\sqrt{d}}{B_0}$ is a quadratic algebraic number, the elements α_k, which are defined inductively by $\alpha_0 = \alpha$, $a_k = [\alpha_k]$ (integer part), and $\alpha_{k+1} = 1/(\alpha_k - a_k)$ can be written as
$$\alpha_k = \frac{A_k + \sqrt{d}}{B_k},$$
where $A_{k+1} = a_k B_k - A_k$ and $B_{k+1} = (d - A_{k+1}^2)/B_k$. Also recall that the convergents $C_k = p_k/q_k$ are defined by $p_0 = a_0$, $p_1 = a_0 a_1 + 1$, $q_0 = 1$ and $q_1 = a_1$, and by the recursion relations
$$p_k = a_k p_{k-1} + p_{k-2}, \qquad q_k = a_k q_{k-1} + q_{k-2}.$$
(a) Show for the special case $\alpha = \sqrt{d}$ that
$$p_k^2 - d q_k^2 = (-1)^{k+1} B_{k+1}, \qquad p_k p_{k-1} - d q_k q_{k-1} = (-1)^{k+1} A_{k+1}.$$
(b) Assume that $\sqrt{d}$ has the periodic continued fraction expansion $[a_0; \overline{a_1, a_2, \ ..., a_m}]$ (i.e. $a_{k+m} = a_k$ for $k > 0$; proof of this on Monday). Show that $p_{k+m}^2 - d q_{k+m}^2 = (-1)^m (p_k^2 - d q_k^2)$ for all $k > 0$.
(c) Prove that there exists a solution of Pell's equation $x^2 - dy^2 = (-1)^m$ of the form $x = p_{m-1}$, $y = q_{m-1}$. ($Hint$: Periodicity only helps to compare α_1 with α_{m+1}; this should still help to figure out B_m).
(d) Prove that $x^2 - dy^2 = -1$ has a solution if and only if the period m for the continued fraction expansion for $\sqrt{d}$ is odd. What about $x^2 - 21y^2 = -1$?