

MATH 110 MIDTERM FALL 2020

Please justify all your steps! This is an open book exam. The precise rules were stated on the course web page. If you have questions, you can email me at *wenzl.hans@gmail.com*.

0. Write on your first exam page *I excel with Integrity*. This means that you adhere to the following principles (you need not write them on your exam):

I am **fair** to my class mates and instructors by not using any unauthorized aids.

I **respect** myself and my university by upholding educational and evaluative goals.

I am **honest** in my representations of myself and my work.

I accept **responsibility** for ensuring my actions are in accord with academic integrity.

I show I am **trustworthy** even when no one is watching.

1. Consider the function  $f(x) = x^2$  on the interval  $[0, \pi]$ .

(a) Calculate the Fourier sine series of  $f(x)$ .

(b) The Fourier cosine series of  $f(x)$  is given by

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos(nx). \quad (*)$$

Calculate the Fourier sine series of  $x = \frac{1}{2}f'(x)$  via term-by-term differentiation from the series above. *Important* : Justify why this is allowed in this case.

(c) Determine the full Fourier series for the function  $f(x) = x^2$  for the interval  $[-\pi, \pi]$  from the calculations above and justify your answer. You may assume the result in (a) to be known regardless whether you could do it or not.

(d) What is the value of the series (\*) in (b) for  $x = \pi$ ? Justify your answer! Use this to calculate  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ .

*Please look at next page for more problems*

2. Find the solution of the wave equation for a string of length  $L = \pi$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad (PDE)$$

$$u(0, t) = 0 = u(\pi, t), \quad (BC)$$

with initial conditions

$$u(x, 0) = 2 \sin 5x, \quad (IC1)$$

$$\frac{\partial u}{\partial t}(x, 0) = x. \quad (IC2)$$

You may use any result we have derived in the lecture or which has been obtained in previous problems in this exam.

3. Consider the 1-dimensional wave equation with  $L = \pi$  and with additional force

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \alpha u, \quad (PDE)$$

$$u(0, t) = 0 = u(\pi, t). \quad (BC)$$

Consider product solutions  $u(x, t) = \Phi(x)h(t)$ .

(a) Find differential equations for  $\Phi(x)$  and  $h(t)$ . *Hint* : It will be convenient to set them up such that  $\Phi''(x) = -\lambda\Phi(x)$ .

(b) Determine the possible eigenvalues  $\lambda$  and eigenfunctions  $\Phi$  subject to the given boundary conditions (*BC*). You may refer to results obtained in class or in the book.

(c) Determine the corresponding functions  $h(t)$  for each eigenvalue  $\lambda$  determined in (b) and write down all possible product solutions.