Solutions from Chapter 9.1 and 9.2

Section 9.1, Problem # 1. This basically boils down to an exercise in the chain rule from calculus. We are looking for solutions of the form:

$$u(t,\vec{x}) \;=\; f(\vec{k}\cdot\vec{x} - ct) \ ,$$

where $\vec{k},\vec{x}\in\mathbb{R}^3$, and $\vec{k}$ is some fixed vector. With this setup, we can easily calculate that:

$$\partial_t u \;=\; -cf' \ , \qquad \partial_x u \;=\; k_x f' \ , \qquad \partial_y u \;=\; k_y f' \ , \qquad \partial_z u \;=\; k_z f' \ .$$

Here we are writing $\vec{k} = (k_x, k_y, k_z)$ for the components of $\vec{k}$ (which are fixed). By simply repeating this computation, we see that:

$$\partial_t^2 u \;=\; c^2 f'' \ , \qquad \partial_x^2 u \;=\; k_x^2 f'' \ , \qquad \partial_y^2 u \;=\; k_y^2 f'' \ , \qquad \partial_z^2 u \;=\; k_z^2 f'' \ .$$

Combining all of this, we have the relation:

$$\partial_t^2 u - c^2\Delta u \;=\; c^2(|\vec{k}|^2 - 1)f'' \ .$$

If we want the expression on the right hand side above to vanish, we simply need to choose either of:

$$|\vec{k}|^2 - 1 \;=\; 0 \ , \qquad\qquad f'' \;=\; 0 \ .$$

Thus, we need either of:

$$|\vec{k}| \;=\; 1 \ , \qquad\qquad f(\xi) \;=\; a\xi + b \ ,$$

where $\xi\in\mathbb{R}$ is a single real variable, and $a,b\in\mathbb{R}$ are fixed constants.

Section 9.2, problem # 5. For this problem, suppose that u solves the three dimensional wave equation:

$$\begin{aligned} \partial_t^2 u - c^2\Delta u \;&=\; 0 \ , \\ u(0,\vec{x}) \;&=\; f(\vec{x}) \ , \\ u_t(0,\vec{x}) \;&=\; g(\vec{x}) \ . \end{aligned}$$

Let us also suppose the initial data f and g are such that they vanish outside of a ball of radius ρ. That is:

$$f(\vec{x}) \;=\; g(\vec{x}) \;\equiv\; 0 \ , \qquad\qquad \text{for} \ \ \rho \leqslant |\vec{x}| \ . \tag{1}$$

From the Kirchhoff formula:

$$u(t,\vec{x}_0) \;=\; \frac{1}{4\pi c^2 t}\iint_{S_{ct}(\vec{x}_0)} g(\vec{x})\ dS_{\vec{x}} \;+\; \partial_t\left[\frac{1}{4\pi c^2 t}\iint_{S_{ct}(\vec{x}_0)} f(\vec{x})\ dS_{\vec{x}}\right] \ ,$$

it is clear that one always has $u(t,\vec{x}_0) = 0$ whenever the point $\vec{x}_0$ is such that the sphere $S_{ct}(\vec{x}_0)$ centered at this point only gives the value zero for $g|_{S_{ct}(\vec{x}_0)}$ and $f|_{S_{ct}(\vec{x}_0)}$. That is, whenever the initial data f, g both vanish on this sphere.

We are now looking for the set of points $(t, \vec{x}_0)$ such that both of the following are true:

$$g|_{S_{ct}(\vec{x}_0)} \equiv 0 , \qquad\qquad f|_{S_{ct}(\vec{x}_0)} \equiv 0 .$$

From the conditions (1), all we need to do is to choose spheres $S_{ct}(\vec{x}_0)$ such that:

$$S_{ct}(\vec{x}_0) \cap \{|\vec{x}| < \rho\} = \emptyset .$$

It is not hard to check that this is accomplished whenever:

$$|\vec{x}_0| \leqslant ct - \rho ,$$

or:

$$ct + \rho \leqslant |\vec{x}_0|.$$

(You should check this for yourself!) Summing up, this means that the solution u must vanish outside of a spherical shell centered a the origin of inner radius $r = ct - \rho$, and outer radius $R = ct + \rho$. Of course for times where $ct - \rho < 0$, the solution just vanished outside of ball of radius R.

Section 9.2, problem # 6. We'll answer these questions in order:

- In this problem, the center point of the sphere of radius R should probably be called $\vec{x}_0$. To do the computation, we'll set everything up in polar coordinates centered at $\vec{x}_0$, and such that in these coordinates the positive z axis contains the line segment between $\vec{x}_0$ and the center of the ball of radius ρ which was the "old" origin. Its not too hard to see that after a translation and rotation (which does not effect things like surface areas) we may reduce to this special case regardless of the original position of $\vec{x}_0$.

  In this configuration, we see the area we are trying to compute is that of a spherical cap on the sphere of radius R centered at the origin, where the boundary circle of the cap is precisely the intersection of the sphere of radius R at $(0,0,0)$ and the ball of radius ρ centered at the point $(0,0,|\vec{x}_0|)$, where $\vec{x}_0$ is the original (before translation) center of the sphere of radius R.

  To compute the area of the spherical cap, we now only need to know the widest angle that this cap makes with the z-axis. If we call this angle ϕ_0, then from basic calculus we see that the area may be computed as:

$$A_{cap} = \int_0^{2\pi} \int_0^{\phi_0} \sin(\phi) R^2 d\phi d\theta . \tag{2}$$

  To compute the angle ϕ_0, we simply use the law of cosines, which says that:

$$\rho^2 = |\vec{x}_0|^2 + R^2 - 2|\vec{x}_0| R \cos(\phi_0) .$$

Notice that we are computing things with respect to the triangle with vertices $(0,0,0)$, $(0,0,|\vec{x}_0|)$, and the point of intersection of our two spheres which lies in the first quadrant of the z-x plane (you may want to draw a picture of all of this to get a feeling for it). Therefore, we see that the angle ϕ_0 is defined implicitly via the relation:

$$-\cos(\phi_0) \;=\; \frac{\rho^2 - |\vec{x}_0| - R^2}{2|\vec{x}_0|R} \;.$$

This easily allows us to compute the integral (2), and the answer is:

$$\begin{aligned} A_{cap} \;&=\; 2\pi R^2 \cdot \Big(-\cos(\phi)\big|_0^{\phi_0} \Big) \;. \\ &=\; 2\pi R^2 \left(\frac{\rho^2 - |\vec{x}_0| - R^2}{2|\vec{x}_0|R} + 1 \right) , \\ &=\; \frac{\pi R\big(\rho^2 - (|\vec{x}_0| - R)^2\big)}{|\vec{x}_0|} \;. \end{aligned} \tag{3}$$

Notice that this last formula is not valid when there is some nontrivial intersection between these two spheres. In the case where $\rho + R < |\vec{x}_0|$ there is no intersection, and therefore no area. On the other hand, in the case where $|\vec{x}_0| + R < \rho$ there is also no intersection, but this time the entire sphere of radius R centered at $\vec{x}_0$ is contained in the ball of radius ρ centered at the origin. Therefore, in this second case the area is just $4\pi R^2$.

- To compute the solution for this specific initial value problem, we use the Kirchhoff formula which in this case says that:

$$u(t, \vec{x}_0) \;=\; \frac{1}{4\pi c^2 t} \iint_{S_{ct}(\vec{x}_0)} A \cdot \chi_\rho(\vec{x}) \; dS_{\vec{x}} \;,$$

where χ_ρ is the cutoff function:

$$\chi(\vec{x}) \;=\; \begin{cases} 1 \;, & \text{where } |\vec{x}| \leqslant \rho \;, \\ 0 \;, & \text{otherwise} \;. \end{cases} \tag{4}$$

Now, this integral is just A times the area of the intersection of the sphere of radius ct centered at $\vec{x}_0$ and the solid ball of radius ρ centered at the origin, which is what we just computed. There are three cases depending on the geometry (as explained at the end of the last paragraph), and we see from using formula (3) that the answer is:

$$u(t, \vec{x}) \;=\; \begin{cases} A \cdot t \;, & \text{if } |\vec{x}| < \rho - ct \;, \\ A \cdot \frac{\big(\rho^2 - (|\vec{x}| - ct)^2\big)}{4c|\vec{x}|} \;, & \text{if } \;\rho - ct \leqslant |\vec{x}| \leqslant \rho + ct \;, \\ 0 \;, & \text{if } \rho + ct < |\vec{x}| \;. \end{cases} \tag{5}$$

- For problems c) and d), its not so easy to type up a picture, so I'll just skip it. The above formula can be used to obtain any information one needs to draw a picture.

- If we compute the solution $u(t, \vec{x}_0 + t\vec{v})$ for t large and $\vec{x}_0$ fixed, with $|\vec{v}| = c$ and $|\vec{x}_0| < \rho$, then we will be in the middle case of formula (5) above. That is, we are trying to compute the limit:

$$
\begin{aligned}
\lim_{t\to\infty} t \cdot u(t, \vec{x}_0 + t\vec{v}) &= \lim_{t\to\infty} A \cdot \frac{\big(\rho^2 - (|\vec{x}_0 + \vec{v}t| - ct)^2\big)}{4ct^{-1}|\vec{x}_0 + t\vec{v}|} \ , \\
&= \lim_{t\to\infty} A \cdot \frac{\big(\rho^2 - (|\vec{x}_0 + \vec{v}t| - ct)^2\big)}{4c|\frac{\vec{x}_0}{t} + \vec{v}|} \ .
\end{aligned} \tag{6}
$$

To complete the computation of the limit, it suffices to be to compute:

$$
\begin{aligned}
\lim_{t\to\infty} (|\vec{x}_0 + \vec{v}t| - ct) &= \lim_{t\to\infty} (|\vec{x}_0 + \vec{v}t| - ct) \cdot \left(\frac{|\vec{x}_0 + \vec{v}t| + ct)}{|\vec{x}_0 + \vec{v}t| + ct)}\right) \\
&= \lim_{t\to\infty} \frac{|\vec{x}_0 + \vec{v}t|^2 - (ct)^2}{|\vec{x}_0 + \vec{v}t| + ct} \ , \\
&= \lim_{t\to\infty} \frac{|\vec{x}_0|^2 + 2t\vec{x}_0 \cdot \vec{v}}{|\vec{x}_0 + \vec{v}t| + ct} \ , \\
&= \lim_{t\to\infty} \frac{\frac{|\vec{x}_0|^2}{t} + 2\vec{x}_0 \cdot \vec{v}}{|\frac{\vec{x}_0}{t} + \vec{v}| + c} \ , \\
&= \frac{\vec{x}_0 \cdot \vec{v}}{c} \ .
\end{aligned}
$$

Plugging this computation into the line (6) above, we have:

$$
\lim_{t\to\infty} t \cdot u(t, \vec{x}_0 + t\vec{v}) = \frac{A}{4c^2}\big(\rho^2 - \frac{(\vec{x}_0 \cdot \vec{v})^2}{c^2}\big) \ .
$$

Section 9.2, Problem # 9. We'll solve each of these separately:

- This problem is almost identical to part of Problem 5 above. In particular, if the initial data f, g both vanish outside of the sphere of radius ρ (centered at the origin), then the corresponding solution u must vanish inside the sphere of radius $R(t) = \rho - ct$, for later times $R(t)$ such that $0 < R(t)$.

- To obtain this uniform behavior, lets just look at the situation where the initial displacement $f(\vec{x})$ is identically zero. In the more general case where it is not zero, one can also prove decay on the order of t^{-1} by essentially the same argument, but one needs to use a change of variables in the Kirchhoff integral formula as we had discussed in recitation.

  We are now trying to show that if $u(t, \vec{x}_0)$ is given by the formula:

$$
u(t, \vec{x}_0) = \frac{1}{4\pi c^2 t}\iint_{S_{ct}(\vec{x}_0)} g(\vec{x}) \, dS_{\vec{x}} \ ,
$$

then one has the uniform estimate:

$$|u(t,\vec{x}_0)| \ \leqslant \ \frac{C}{t} \ .$$

To prove this, it clearly suffices to show that one simply has a uniform bound on the integral in the Kirchhoff formula. That is, we only need to show that:

$$\left| \iint_{S_{ct}(\vec{x}_0)} g(\vec{x}) \ dS_{\vec{x}} \right| \ \leqslant \ C \ , \tag{7}$$

for some constant C which will depend on g, but not on the point $(t,\vec{x}_0)$. To prove this bound, we'll need to use the following basic estimate from analysis:

$$\left| \iint_{S_{ct}(\vec{x}_0)} g(\vec{x}) \ dS_{\vec{x}} \right| \ \leqslant \ \iint_{S_{ct}(\vec{x}_0)} |g(\vec{x})| \ dS_{\vec{x}} \ , \tag{8}$$

where the bound holds for any function where the integrals converge. This last bound reduces our task to estimating some basic integrals, because by the conditions we are placing on the initial data g, its not too difficult to see that:

$$|g(\vec{x})| \ \leqslant \ M \cdot \chi_\rho(\vec{x}) \ ,$$

where $M = \max|g|$ and χ_ρ is the radial cutoff function defined on line (4) above. Here ρ is chosen so that $g(\vec{x}) \equiv 0$ whenever $\rho \leqslant |\vec{x}|$. Using this notation, the bound (8) implies that:

$$\left| \iint_{S_{ct}(\vec{x}_0)} g(\vec{x}) \ dS_{\vec{x}} \right| \ \leqslant \ M \cdot \iint_{S_{ct}(\vec{x}_0)} \chi_\rho(\vec{x}) \ dS_{\vec{x}} \ .$$

Therefore, (by changing the definition of the constant C) we see that to establish (7) we only have to prove the bound:

$$\iint_{S_{ct}(\vec{x}_0)} \chi_\rho(\vec{x}) \ dS_{\vec{x}} \ \leqslant \ C \ , \tag{9}$$

where C is a uniform constant that only depends on ρ (and therefore on the properties of g), but *not* on the point $(t,\vec{x}_0)$. The proof of (9) is quite easy if one does not insist on being too precise. The point is that its more or less geometrically obvious that the sphere of radius ct centered at the point $\vec{x}_0$ only intersects the ball of radius ρ in a surface patch of area uniformly bounded by a constant depending only on ρ. I'll leave the details of this to the reader (i.e. see if you can come up with a precise estimate).