6.2.2) \[ \dot{x} = y \quad \dot{y} = -x + (1 - x^2 - y^2)y \]

a) It's clear that both \( \dot{x} \) and \( \dot{y} \) are continuous on the open disk \( x^2 + y^2 < 1 \) as they're polynomials and so are their partial derivatives \( \dot{x} \) and \( \dot{y} \) are continuous as well since they're polynomials. So the uniqueness and existence theorem is satisfied. [1 point]

b) Using \( x(t) = \sin t \) and \( y(t) = \cos t \), show that \( \dot{x}(t) = \cos t \) and \( \dot{y}(t) = -\sin t \).

\[ \dot{x}(t) = y(t) = \cos t \quad \text{and} \quad \dot{y}(t) = -x + (1 - x^2 - y^2)y \]

\[ \implies \dot{y}(t) = -\sin t + (1 - \sin^2 t - \cos^2 t) \cos t \]

\[ = -\sin t + (1 - \left(\cos^2 t + \sin^2 t\right)) \cos t = -\sin t \] [1 point]

c) From b) since \( x(t) = \sin t \) and \( y(t) = \cos t \) are solutions we know this forms the equation for a closed orbit \( (x(t)^2 + y(t)^2 = 1 = \text{const.} \) which encloses the point \((\frac{1}{2}, 0)\). From the uniqueness theorem trajectories cannot intersect and so if \( x(0) = \frac{1}{2} \) and \( y(0) = 0 \) it follows the trajectory must be somewhere \( x^2 + y^2 < 1 \). [1 point]
6.3.2) \( \dot{x} = \sin y \quad \dot{y} = x - x^3 \)

Find fixed points: 
\[ \dot{x} = 0 = \sin y \Rightarrow y = n\pi \quad \text{where } n \in \mathbb{Z} \]
\[ \dot{y} = 0 = x - x^3 \Rightarrow x = 0, \pm 1 \]  
[1 point for fixed points]

Linearize around the fixed points using the Jacobian:

\[ A(x, y) = \begin{bmatrix} 0 & \cos y \\ 1 - 3x^2 & 0 \end{bmatrix} \quad A(0, 0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

\[ A(\pm 1, 0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \quad A(0, \pi) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot A(\pm 1, \pi) = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \]

(Almost y = 0, y = \pi are sufficient as further y values are repetitive)

Classify fixed points:

For 
\( (0, 0) \quad \lambda = \pm 1 \Rightarrow \text{saddle point} \quad \pm \text{vector}\)

\( (\pm 1, 0) \quad \lambda = \pm \sqrt{2} i \Rightarrow \text{center} \quad \pm i \)

\( (0, \pi) \quad \lambda = \pm i \Rightarrow \text{center} \quad \pm \text{i vector}\)

\( (\pm 1, \pi) \quad \lambda = \pm \sqrt{2} \Rightarrow \text{saddle point} \quad \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{2}}{2} \)

\( (\pm 1, -\pi) \quad \lambda = \pm \sqrt{2} \Rightarrow \text{saddle} \quad \pm \frac{\sqrt{2}}{2}, \pm \frac{i}{2} \)

Nullclines:
- horizontal: \( \dot{y} = 0 \Rightarrow x = 0, \pm 1 \)
- vertical: \( \dot{x} = 0 \Rightarrow y = n\pi \)
Using all of the information we draw the phase portrait.

Total: 7 points
6.5.1) \[ \dot{x} = x^3 - x \]

a) \[ \dot{x} = y, \quad \dot{y} = x^2 - x \]

\[ \dot{x} = 0 \Rightarrow y = 0 \quad \dot{y} = 0 = x^3 - x \Rightarrow x = 0, \pm 1 \]

Linearize around fixed points and find eigenvalues and eigenvectors.

\[ A(x, y) = \begin{bmatrix} 0 & 1 \\ 3x^2 & 0 \end{bmatrix} \quad A(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad A \pm 1 \pm 0 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \]

\[ \lambda = \pm i \quad \text{Center} \]
\[ \lambda = \pm \sqrt{2} \quad \text{Saddle point} \]
\[ (\sqrt{2}, \sqrt{2}) \]

[1 point for each classification]

b) One approach to this is to consider the system as a force and potential system with \( F(x) = \dot{x} = -\frac{dv}{dx} \)

\[ \Rightarrow V(x) = \int x^2 - x^4 \, dx = \frac{x^3}{2} - \frac{x^5}{4} \]

Using the equation for total energy \( E = \frac{1}{2} m \dot{x}^2 + V(x) \)

\[ \Rightarrow E = \frac{1}{2} y^2 + \frac{1}{2} x^2 - \frac{1}{4} x^4 \]

is a conserved quantity (can check by taking time derivatives)

[1 point for correct quantity]
nullclines: \[ \text{vert} - y = 0 \]
\[ \text{horiz} - x = 0, x = \pm 1 \]

[1 point for drawing all fixed points and 1 point for approximate trajectories]

Total: 5 points