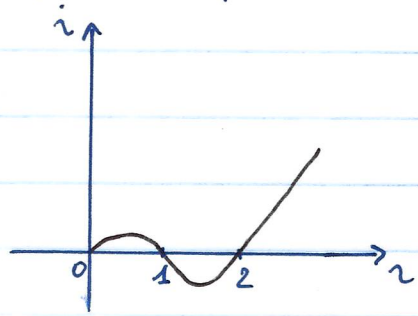
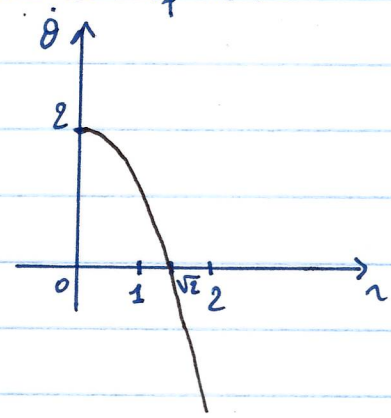
BOB

# Homework 1

**7.1.3:** $\begin{cases} \dot{r} = r(1-r^2)(4-r^2) \\ \dot{\theta} = 2 - r^2 \end{cases}$

First, we can plot $\dot{r}$ and $\dot{\theta}$ in function of $r$:




From this (or directly from the equations) we get that

(i) $\dot{r} > 0 \iff r \in (0,1) \cup (2,\infty)$ ← here trajectories move outward

$\dot{r} < 0 \iff r \in (1,2)$ ← — " — inward

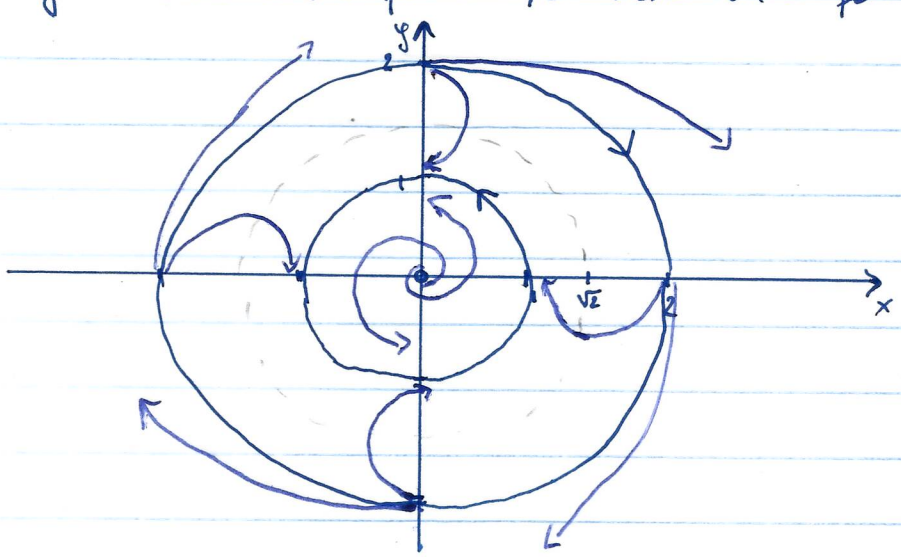
$\dot{r} = 0 \iff r \in \{0,1,2\}$

(ii) $\dot{\theta} > 0 \iff r \in [0,\sqrt{2})$ ← trajectories move counterclockwise

$\dot{\theta} < 0 \iff r \in (\sqrt{2},\infty)$ ← — " — clockwise

$\dot{\theta} = 0 \iff r = \sqrt{2}$

We see that $(0,0)$ is a fixed point. Also the circles $r=1$ and $r=2$ will be limit cycles since $\dot{\theta} \neq 0$ for $r = 1,2$ and $\dot{r} \neq 0$ for $r \neq 0,1,2$. We get:



**7.1.8** $\ddot{x} + a\dot{x}(x^2+\dot{x}^2-1) + x = 0 \quad , a>0$

a) We rewrite the system: $\begin{cases} \dot{x} = y \\ \dot{y} = -ay(x^2+y^2-1) - x \end{cases}$

Putting both equal to zero, we find as only fixed point $(x^*, y^*) = (0,0)$.
The Jacobian at $(0,0)$ is $\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$, in particular $\tau = a$, $\Delta = 1$.

From chapters 5 & 6: if $a<2$ : unstable spiral
$a>2$ : unstable node
$a=2$ : unstable (but not sure which one, as this is a borderline case).

b) Changing to polar coordinates, one gets

$\dot{r} = \frac{x\dot{x}+y\dot{y}}{r} = \dots = -ar\sin^2\theta(r^2-1)$

$\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2} = \dots = -a\cos\theta\sin\theta(r^2-1) - 1$

So for $r=1$, we get $\dot{r}=0$, $\dot{\theta}=-1$. Hence the circle $r=1$ is a limit cycle with amplitude 1 and period $T = \int_0^{2\pi} \frac{1}{\dot{\theta}}\, d\theta = \int_0^{2\pi} -1\, d\theta = -2\pi$, i.e. period $2\pi$ in clockwise direction.

c) Since $\begin{cases} \dot{r} > 0 & \text{if } r<1 \\ \dot{r} < 0 & \text{if } r>1 \end{cases}$, the limit cycle is stable.

d) Same as c): if $r<1$, the radius always increases and if $r>1$, the radius always decreases, so there are no possibilities for other limit cycles.

**7.2.8**: Suppose $\begin{cases} \dot{x} = -\frac{\partial V}{\partial x} \\ \dot{y} = -\frac{\partial V}{\partial y} \end{cases}$ is a gradient system. Since $V$ is a differentiable function of two variables, its equipotentials are differentiable paths.
Suppose $(\tilde{x}(t), \tilde{y}(t))$ is such an equipotential, i.e. $V(\tilde{x}(t), \tilde{y}(t)) = c$.
Differentiating we get by the chain rule

$0 = \frac{d}{dt} V(\tilde{x}(t), \tilde{y}(t)) = \frac{\partial V}{\partial x}\dot{\tilde{x}} + \frac{\partial V}{\partial y}\dot{\tilde{y}} = -\dot{x}\dot{\tilde{x}} - \dot{y}\dot{\tilde{y}}$ and so the inner product between the tangent to the trajectory $(x,y)$ and the equipotential $(\tilde{x},\tilde{y})$ is 0, i.e. they cross orthogonaly. $\square$