For Dulac's criterion, we need to calculate $\nabla \cdot (g\dot{x})$ and check it doesn't change sign. Here $\dot{x} = (N_1, N_2)$

$$\nabla \cdot (g\dot{x}) = \frac{\partial}{\partial N_1} (gN_1) + \frac{\partial}{\partial N_2} (gN_2) = \frac{\partial}{\partial N_1} \left( \frac{N_1}{N_1} (1 - \frac{N_1}{K_1}) - b_1 \right) + \frac{\partial}{\partial N_2} \left( \frac{N_2}{N_2} (1 - \frac{N_2}{K_2}) - b_2 \right)$$

$$= -\frac{N_1}{N_2 K_1} - \frac{N_2}{N_1 K_2} < 0 \text{ for } N_1, N_2 > 0 \text{ (since it is given that } \tau_1, \tau_2, K_1, K_2 > 0)$$

So for $N_1, N_2 > 0$, there are no periodic orbits.

7.3.3 \[ \dot{x} = x - y - x^3 \quad \dot{y} = x + y - y^3 \]

Consider the square $[-2,2] \times [-2,2]$: we claim this is a trapping region. Indeed, on the boundary we get:

- $x = 2 : \dot{x} = -y - 6 < 0$ for $-2 \leq y \leq 2$
- $x = -2 : \dot{x} = -y + 6 > 0$ for $-2 \leq y \leq 2$
- $y = 2 : \dot{y} = x - 6 < 0$ for $-2 \leq x \leq 2$
- $y = -2 : \dot{y} = x + 6 > 0$ for $-2 \leq x \leq 2$

So the vectors point inward on the whole boundary. Hence if we show that the only fixed point, namely $(0,0)$, is a repellor (unstable), we get the existence of a limit cycle using Poincaré-Bendixson Thm. But for $(0,0)$, we have $A = \begin{pmatrix} 1 & 3x \\ 0 & 1 \end{pmatrix}$, so $\tau = A = 2$ and $(0,0)$ is unstable.
\[ 7.3.4. \begin{align*}
    \dot{x} &= x(1-4x^2-y^2) - \frac{1}{2} y(1+x) \\
    \dot{y} &= y(1-4x^2-y^2) + 2x(1+x)
\end{align*} \]

a) Obviously, the origin is a fixed point. The Jacobian is equal to
\[
A = \begin{pmatrix}
    1 - 4x^2 - y^2 + x(-8x) - \frac{1}{2} y & -2xy - \frac{1}{2} (1+x) \\
    -8xy + 2(1+x) + 2x & 1 - 4x^2 - y^2 + y(-2y)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
    1 & -\frac{1}{2} \\
    2 & 1
\end{pmatrix}
\]

So \( \tau = \Delta = 2 \) and the origin is unstable.

b) Given \( V = (1-4x^2-y^2)^2 \), we get
\[
\dot{V} = 2(1-4x^2-y^2)(-8x\dot{x}+2y\dot{y})
\]

\[
= -4(1-4x^2-y^2)(4x(1-4x^2-y^2) - \frac{1}{2} y(1+x)) + y(y(1-4x^2-y^2) + 2x(1+x))
\]

\[
= -4(1-4x^2-y^2)(1-4x^2-y^2) + 2xy(1+x) + 2y(1+x)
\]

\[
= -4(1-4x^2-y^2)(4x^2+y^2).
\]

We see that \( \dot{V} = 0 \) if \((x, y) = (0, 0)\) or \(-4x^2-y^2 = 0\)
\( \dot{V} < 0 \) otherwise.

So if we are on the ellipse \( 4x^2+y^2 = 1 \), \( V \) remains constant. On all other trajectories (except at the fixed point \((0,0)\)) \( V \) strictly decreases. In particular, there are no periodic orbits outside the ellipse and since by (a1 \((0,0)\) is unstable, all other trajectories have to converge to the ellipse \( 4x^2+y^2 = 1 \) (since they cannot converge to \((0,0)\)).