130B

Homework 2

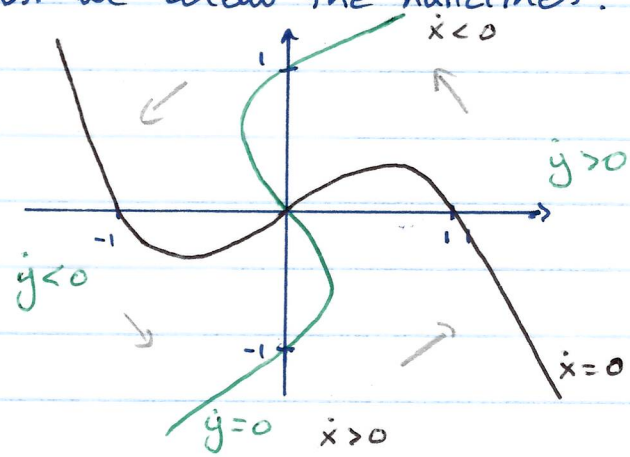
7.2.13 For Dulac's criterion, we need to calculate $\nabla\cdot(g\dot{\vec{x}})$ and check it doesn't change sign. Here $\vec{x}=(N_1, N_2)$

$$\nabla\cdot(g\dot{\vec{x}}) = \frac{\partial}{\partial N_1}(g\dot{N}_1) + \frac{\partial}{\partial N_2}(g\dot{N}_2) = \frac{\partial}{\partial N_1}\left(\frac{r_1}{N_2}\left(1-\frac{N_1}{K_1}\right)-b_1\right) + \frac{\partial}{\partial N_2}\left(\frac{r_2}{N_1}\left(1-\frac{N_2}{K_2}\right)-b_2\right)$$

$$= -\frac{r_1}{N_2K_1} - \frac{r_2}{N_1K_2} < 0 \quad \text{for } N_1, N_2 > 0$$ (since it is given that $r_1, r_2, K_1, K_2 > 0$)

So for $N_1, N_2 > 0$, there are no ~~limit cycles~~ periodic orbits.

7.3.3 $\begin{cases} \dot{x} = x - y - x^3 \\ \dot{y} = x + y - y^3 \end{cases}$. First we draw the nullclines:



Consider the square $[-2,2]\times[-2,2]$: we claim this is a trapping region. Indeed, on the boundary we get:

- $x=2$: $\dot{x} = -y-6 < 0$ for $-2 \le y \le 2$
- $x=-2$: $\dot{x} = -y+6 > 0$ for $-2 \le y \le 2$
- $y=2$: $\dot{y} = x-6 < 0$ for $-2 \le x \le 2$
- $y=-2$: $\dot{y} = x+6 > 0$ for $-2 \le x \le 2$.

So the vectors point inward on the whole boundary. Hence if we show that the only fixed point, namely $(0,0)$, is a repeller (unstable), we get the existence of a limit cycle using Poincaré-Bendixson thm.

But for $(0,0)$, we have $A = \begin{pmatrix} 1-3x^2 & -1 \\ 1 & 1-3y^2 \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, so $\tau = \Delta = 2$ and $(0,0)$ is unstable. $\square$

7.3.4. $\begin{cases} \dot{x} = x(1-4x^2-y^2) - \frac{1}{2}y(1+x) \\ \dot{y} = y(1-4x^2-y^2) + 2x(1+x) \end{cases}$

a) Obviously the origin is a fixed point. The Jacobian is equal to

$$A = \begin{pmatrix} 1-4x^2-y^2+x(-8x)-\frac{1}{2}y & -2xy-\frac{1}{2}(1+x) \\ -8xy+2(1+x)+2x & 1-4x^2-y^2+y(-2y) \end{pmatrix}\Bigg|_{(0,0)}$$

$$= \begin{pmatrix} 1 & -1/2 \\ 2 & 1 \end{pmatrix}$$

So $\tau = \Delta = 2$ and the origin is unstable.

b) Given $V = (1-4x^2-y^2)^2$, we get

$$\dot{V} = 2(1-4x^2-y^2)(-8x\dot{x} - 2y\dot{y})$$

$$= -4(1-4x^2-y^2)\left(4x\left(x(1-4x^2-y^2) - \tfrac{1}{2}y(1+x)\right) + y\left(y(1-4x^2-y^2)+2x(1+x)\right)\right)$$

$$= -4(1-4x^2-y^2)\left((1-4x^2-y^2)(4x^2+y^2) \cancel{- 2xy(1+x)} + \cancel{2yx(1+x)}\right)$$

$$= -4(1-4x^2-y^2)^2(4x^2+y^2).$$

We see that $\begin{cases} \dot{V} = 0 & \text{if } (x,y) = (0,0) \text{ or } 1-4x^2-y^2 = 0 \\ \dot{V} < 0 & \text{otherwise} \end{cases}$

So if we are on the ellipse $4x^2+y^2 = 1$, V remains constant. On all other trajectories (except at the fixed point $(0,0)$) V strictly decreases. In particular there are no periodic orbits outside the ellipse and since by (a) $(0,0)$ is unstable, all other trajectories have to converge to the ellipse $4x^2+y^2=1$ (since they cannot converge to $(0,0)$.).