BOB

Homework 7

9.2.2. We will use a Liapunov-like idea here.
Let $V = rx^2 + \sigma y^2 + \sigma(z-2r)^2$ (the defining function of E)
Then $\dot{V} = 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma(z-2r)\dot{z}$

$= \dots$

$= -2r\sigma x^2 - 2\sigma y^2 - 2b\sigma(z-r)^2 + 2b\sigma r^2$

Hence we see that $\dot{V} < 0$ if $\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} > 1 \left(= \frac{2b\sigma r^2}{2b\sigma r^2}\right)$
Since also $V > 0$ (except at $(0,0,2r)$), we can choose C big enough such that $\frac{x^2}{br} + \frac{y^2}{br^2} + \frac{(z-r)^2}{r^2} = 1$ is contained in E and by the above all trajectories will eventually end up in E and be trapped there forever.

9.2.6 a) We know that $\dot{V} = \int_V \nabla \cdot f \, dV$ where V is the volume.
Since in this case $\nabla \cdot f = -2v < 0$, volume always decreases.

b) The fixed points satisfy

$$\begin{cases} \dot{x} = -vx + zy = 0 & (1) \\ \dot{y} = -vy + (z-a)x = 0 & (2) \\ \dot{z} = 1 - xy = 0 & (3) \end{cases}$$

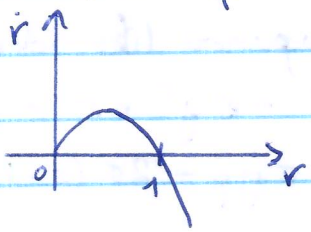
$\overset{(3)}{\Rightarrow} y = \frac{1}{x}$ & $\overset{(1)}{\Rightarrow} -vx + \frac{z}{x} = 0$, i.e. $z = vx^2$
Plugging into (2) gives $-v\cdot\frac{1}{x} + (vx^2 - a)x = 0$
$\Rightarrow v(x^2 - x^{-2}) = a$

Hence the parametric form given.

c) We get the Jacobian $Jac_{(k,k^{-1},vk^2)} = \begin{pmatrix} -v & vk^2 & k^{-1} \\ vk^2 - a & -v & k \\ -k^{-1} & -k & 0 \end{pmatrix}$. ($(-k,-k^{-1},vk^2)$ is similar)

One calculates its characteristic polynomial to find $P(\lambda) = (\lambda + 2v)(\lambda^2 + \frac{a}{v})$. Hence the eigenvalues are $\lambda^* = -2v, \pm i\frac{a}{v}$.
So in one direction the fixed point is stable ($\lambda^* = -2v < 0$) and the other two are borderline cases, so we cannot conclude anything.

9.3.8: a) Yes: if $r=1$, $\dot{r}=0$ and we get a periodic orbit (as $\dot{\vartheta}=1$). Trajectories cannot cross, so if we start inside ($r<1$), we stay inside



b) Yes: if $r>1$, $\dot{r}<0$; so D attracts all initial conditions.

c) No: it's not minimal, $x^2+y^2=1$ is a proper subset which also satisfies a) and b). (see d)

d) Yes: · it's invariant as $\dot{r}=0$ for $r=1$
· it attracts $\mathbb{R}^2 \setminus \{0,0\}$
· it's minimal since it consists of one single orbit.

↳ Its basin of attraction is $\mathbb{R}^2 \setminus \{0,0\}$. Obviously $(0,0)$ is not attracted as it's a fixed point. For $0<r<1$, $\dot{r}>0$ and for $r>1$, $\dot{r}<0$ so it will attract everything else.