MATH 202 FINAL WINTER 2009

- *One cheat sheet allowed, no books or calculators. Please ask if in doubt with any of the notation.*
- *You can use the results of one problem for the solution of another one regardless whether you could solve the former one.*

1. Let G be the group of order 42 with generators t and s, and with relations $t^7 = s^6 = 1$ and $sts^{-1} = t^3$. Determine the structure of its group ring $\mathbf{C}G$. Justify your answer. (You may use the identity $1 + q + q^2 + \ \dots\ + q^{n-1} = 0$ for any $q \neq 1$ satisfying $q^n = 1$, if necessary.)

2. Let $\pi \in S_n$ and let $f_\pi = \#\{i, \pi(i) = i\}$ be the number of its fixed points.
(a) Calculate $\frac{1}{n!}\sum_{\pi\in S_n} f_\pi$.
(b) Calculate $\frac{1}{n!}\sum_{\pi\in S_n} f_\pi^2$.
(c) Show: The sum $\frac{1}{n!}\sum_{\pi\in S_n} f_\pi^m$ is an integer for any positive integer m.

3. Let $V \subset \mathbf{C}G$ be a G submodule of the left regular representation of G. Show that there exists an idempotent $p \in \mathbf{C}G$ such that $V = \mathbf{C}Gp$.

4. Let $P_m(x_1, \ \dots,\ x_N) = x_1^m + \ \dots\ + x_N^m$, and define $P_\lambda = \prod_{j=1}^k P_{\lambda_j}$, where k is the number of rows of λ.
(a) Show that P_λ is homogeneous of degree $|\lambda| = \sum_i \lambda_i$.
(b) Write the power symmetric function P_m as a linear combination of Schur functions.
(c) Give a reason why for any finite group G the matrix $(\chi_\lambda(\mathbf{g}))$, with λ running through the simple representations of G and $\mathbf{g}$ running through the conjugacy classes of G is invertible (Don't spend too much time if you do not remember the argument).
(d) Show that the functions $(P_\lambda)_{|\lambda|=n}$ form a basis for homogeneous symmetric functions in the variables $x_1, x_2, \ \dots, x_N$ of degree n for $n \leq N$.

5. Let G be a finite group, and let H be a subgroup of index 2. Recall that in this case we have $gH = Hg$ for all $g \in G$. Let χ be a simple character of G.
(a) Show that the restriction of χ to H is NOT simple if and only if $\chi(g) = 0$ for all $g \notin H$.
(b) Let V be the G-module corresponding to χ as in (a). How does it decompose as an H-module? (i.e. how many simple H-modules, how many of them isomorphic.)

6. Let $\lambda = [3, 1, 1]$ and let A_5 be the subgroup of all $\pi \in S_5$ for which $\epsilon(\pi) = 1$.
(a) How does the simple S_5-module S^λ decompose as an A_5 module?
(b) Calculate the dimensions of all simple S_5-modules. (You should only use whatever has been PROVED in class and/or homeworks).
(c) Find a complete set of simple representations of A_5 up to isomorphism. (This would mean that you have to show that they are simple, mutually nonisomorphic, and every other simple A_5 module would have to be isomorphic to one of them). Again, you should only use what has been proved so far in class and/or homework.