SOLUTION SKETCHES: MATH 202 FINAL WINTER 2009

The following are, I hope, fairly detailed outlines for the solutions of the problems in the final. Due to time limitations, some of the solutions given here will be sketchier than I would ideally want them in an exam. Please do not be shy about asking if you find some confusing or unclear arguments.

1. Let G be the group of order 42 with generators t and s, and with relations $t^7 = s^6 = 1$ and $sts^{-1} = t^3$. Determine the structure of its group ring $\mathbf{C}G$. Justify your answer. (You may use the identity $1 + q + q^2 + \ ... \ + q^{n-1} = 0$ for any $q \neq 1$ satisfying $q^n = 1$, if necessary.)

   Solution 1 : We consider induced representations from the subgroup T generated by t. Let $\theta = e^{2\pi i/7}$ and let V be the one-dimensional T module determined by $t.v = \theta v$ for $v \in V$. Observe that the powers of s form a set of representatives of the cosets of T in G; indeed it follows from the group relations that any group element can be written in the form $s^j t^r$. Then we have

$$\mathrm{Ind}\chi_V(t) \ = \ \sum_{j=1}^{6} \chi_V(s^j t s^{-j}) \ = \ \theta + \theta^3 + \theta^2 + \theta^{-1} + \theta^{-3} + \theta^{-2},$$

   where we calculated $s^j t s^{-j} = s(s^{j-1} t s^{1-j}) s^{-1}$ inductively. Observe that this sum is equal to -1, using the hint. We obtain the character of t^r by replacing θ by θ^r in the sum above. If $7 \not| r$, this just permutes the seventh roots of unity which are not equal to 1 (ideally there should be a few more details here). Hence we also get $\mathrm{Ind}\chi_V(t^r) = -1$ for all $0 < r < 7$. If $g = s^m t^r$ with $s^m \neq 1$, we also have $s^j g s^{-j} = s^m s^j t^r s^{-j} = s^m t^{r'}$ for some r'. Hence also $s^j g s^{-j} \notin T$ for all $j \in \mathbf{N}$ and therefore

$$\mathrm{Ind}\chi_V(g) \ = \ \sum_{j=1}^{6} \chi_V(s^j g s^{-j}) \ = 0.$$

   Finally, we obviously have $\mathrm{Ind}\chi_V(1) = 6$. Hence we get

$$\langle \mathrm{Ind}\chi_V, \mathrm{Ind}\chi_V \rangle \ = \frac{1}{42}(6^2 + 6(-1)^2 + 35(0)^2) = 1.$$

   Hence the induced representation is simple, i.e. we have a simple 6-dimensional representation of G. Moreover, there are also six one-dimensional representations (see Solution 2). Hence we obtain $\mathbf{C}G \cong \mathbf{C}^6 \oplus M_6(\mathbf{C})$.
   Solution 2 : The idea is to calculate the number of conjugacy classes of the group and guess enough simple representations to deduce the structure of the group ring.
   Conjugacy classes: We have the conjugacy classes $\{1\}$ and

$$\{t, t^3, t^9 = t^2, t^6 = t^{-1}, t^{-3} = t^4, t^{12} = t^5\},$$

where the second one was obtained by conjugating t by s repeatedly. To calculate the remaining conjugacy classes, observe that if a conjugacy class contains $s^j t$ with $0 < j < 6$, then it also contains $s^j t^m$, with $0 < m < 7$ by the same argument as before (i.e. conjugating by s repeatedly). This conjugacy class also needs to contain s^j which can be seen as follows. We also have $ts^j t^{-1} = (tst^{-1})^j = (st^2)^j$, and the latter is equal to $s^j t^r$ for some r. If $t^r = 1$, then t would commute with s^j. But as we have seen in the calculation for our second conjugacy class, this is the case only if $s^j = 1$. *Result*: We have altogether seven conjugacy classes, with the remaining five conjugacy classes being of the form $\{s^j t^r, 0 \leq r < 7\}$ for $1 \leq j < 6$.
Group ring: It follows immediately that we have six one-dimensional representations of G, determined by $s \mapsto e^{2k\pi i/6}$ amd $t \mapsto 1$, with $0 \leq k < 6$. They are obviously non-equivalent (as the characters of s are all non-equal). As we only have seven conjugacy classes, we therefore only have one more simple representation, up to isomorphism, of dimension d. Moreover, we have $42 = \dim \mathbf{C}G = 6 \cdot 1^2 + d^2$, from which we deduce $d = 6$. Hence $\mathbf{C}G \cong \mathbf{C}^6 \oplus M_6(\mathbf{C})$.

2. Let $\pi \in S_n$ and let $f_\pi = \#\{i, \pi(i) = i\}$ be the number of its fixed points.
(a) Calculate $\frac{1}{n!} \sum_{\pi \in S_n} f_\pi$.
(b) Calculate $\frac{1}{n!} \sum_{\pi \in S_n} f_\pi^2$.
(c) Show: The sum $\frac{1}{n!} \sum_{\pi \in S_n} f_\pi^m$ is an integer for any positive integer m.

Solution It was shown in class that the character χ of the natural n-dimensional representation V of S_n is given by $\chi(\pi) = f_\pi$. Let χ_0 be the trivial character of S_n. Then we have

$$\frac{1}{n!} \sum_{\pi \in S_n} f_\pi = \langle \chi, \chi_0 \rangle = 1,$$

where the last equality follows from the fact that $\chi = \chi_0 + \chi_{[n-1,1]}$ is the sum of two simple characters, one of them equal to the trivial character (This fact was shown in class). This shows (a). Similarly, we have

$$\frac{1}{n!} \sum_{\pi \in S_n} f_\pi^2 \;=\; \langle \chi, \chi \rangle \;=\; 2,$$

again using the fact that $\chi = \chi_0 + \chi_{[n-1,1]}$ and that the simple characters form an orthonormal basis with respect to the standard bilinear form $\langle\ ,\ \rangle$ on class functions. To prove (c), observe that $f_\pi^m = \chi_V(\pi)^m = \chi_{V^{\otimes m}}(\pi)$ Hence we have

$$\frac{1}{n!} \sum_{\pi \in S_n} f_\pi^m \;=\; \langle \chi_{V^{\otimes m}}, \chi_0 \rangle = k,$$

where k is the multiplicity of the trivial representation in $V^{\otimes m}$. This is obviously an integer.

3. Let $V \subset \mathbf{C}G$ be a G submodule of the left regular representation of G. Show that there exists an idempotent $p \in \mathbf{C}G$ such that $V = \mathbf{C}Gp$.

Solution We know that $\mathbf{C}G$ is semisimple as a G-module. Hence there exists a G-submodule $W \subset \mathbf{C}G$ such that $\mathbf{C}G = V \oplus W$. Let P be the projection onto V with kernel W. Then P commutes with the G action, i.e. $P \in \mathrm{End}_G(\mathbf{C}G)$. It was shown in class that $\mathrm{End}_G(\mathbf{C}G)$ coincides with $\mathbf{C}G$ acting from the right on itself. Hence there exists $p \in \mathbf{C}G$ such that $Pu = up$ for any $u \in \mathbf{C}G$. Moreover, we have

$$p^2 = 1p^2 = P^2(1) = P(1) = 1p = p.$$

Hence p is an idempotent.

4. Let $P_m(x_1, \ ..., \ x_N) = x_1^m + \ ... \ + x_N^m$, and define $P_\lambda = \prod_{j=1}^k P_{\lambda_j}$, where k is the number of rows of λ.
(a) Show that P_λ is homogeneous of degree $|\lambda| = \sum_i \lambda_i$.
(b) Write the power symmetric function P_m as a linear combination of Schur functions.
(c) Give a reason why for any finite group G the matrix $(\chi_\lambda(\mathbf{g}))$, with λ running through the simple representations of G and $\mathbf{g}$ running through the conjugacy classes of G is invertible (Don't spend too much time if you do not remember the argument).
(d) Show that the functions $(P_\lambda)_{|\lambda|=n}$ form a basis for homogeneous symmetric functions in the variables $x_1, x_2, \ ..., x_N$ of degree n for $n \leq N$.

Solution: P_m is homogeneous of degree m. P_λ is a product of homogeneous polynomials. Hence it must be homogeneous itself. Formal proof can be done by induction on the number of factors in a product of homogeneous polynomials. This shows (a). It follows from Frobenius' formula that $P_m = \sum_\lambda \chi_\lambda(\pi) s_\lambda$, where $\pi \in S_m$ is an m-cycle and χ_λ is the simple character belonging to the partition λ. It was shown in class that $\chi_\lambda(\pi) = 0$ unless λ is a hook. If λ is a hook with r rows, we have $\chi_\lambda(\pi) = (-1)^{r-1}$. To show (c), we use the fact that the simple characters form an orthonormal basis of class functions of G. In particular, they are linearly independent. Now if the determinant were zero, we could find scalars a_χ, not all of them equal to zero, such that $\sum_\chi a_\chi \chi(g) = 0$ for all $g \in G$ (i.e. the columns of matrix $(\chi(g))$ labeled by the characters of G are linearly dependent). But this implies that the characters themselves are linearly dependent, a contradiction.
Finally, to show (d), let π_λ be a permutation in S_n which has cycles of length λ_1, λ_2, ... (i.e. if λ has m rows of length λ_i, π_λ has m λ_i-cycles). Then it follows from Frobenius' formula that

$$P_\lambda = \sum_\mu \chi_\mu(\pi_\lambda) s_\mu.$$

This means the P_λ's are expressed as linear combinations of the Schur functions s_μ's with the coefficients given by the entries of the character matrix for S_n, if $|\lambda| = n$. By (c), this matrix is invertible. Hence also the P_λ with $|\lambda| = n$ are linearly independent. As the space of homogeneous symmetric functions of degree n in $N \geq n$ variables has as dimension the number of Young diagrams with n boxes, the P_λ's with $|\lambda| = n$ also span it, i.e. they form a basis.

5. Let G be a finite group, and let H be a subgroup of index 2. Recall that in this case we have $gH = Hg$ for all $g \in G$. Let χ be a simple character of G.
(a) Show that the restriction of χ to H is NOT simple if and only if $\chi(g) = 0$ for all $g \notin H$.
(b) Let V be the G-module corresponding to χ as in (a). How does it decompose as an H-module? (i.e. how many simple H-modules, how many of them isomorphic.)

Solution: Let χ_H be the restriction of χ to H. Observe that

$$1 = \langle \chi, \chi \rangle_G = \frac{1}{|G|}(\sum_{g \in H} \chi(g)\bar{\chi}(g) + \sum_{g \notin H} \chi(g)\bar{\chi}(g)) = \frac{1}{2}\langle \chi_H, \chi_H \rangle + \frac{1}{|G|} \sum_{g \notin H} \chi(g)\bar{\chi}(g).$$

Hence if $\chi(g) = 0$ for all $g \notin H$, $\langle \chi_H, \chi_H \rangle = 2$. On the other hand, if $\chi(g) \neq 0$ for some $g \notin H$, we have $\sum_{g \notin H} \chi(g)\bar{\chi}(g)) > 0$ and hence $\frac{1}{2}\langle \chi_H, \chi_H \rangle < 1$. This forces $\langle \chi_H, \chi_H \rangle = 1$. For (b), it follows that if V is not a simple H-module, we have $\langle \chi_H, \chi_H \rangle = 2$, by (a). Hence if $V = \oplus V_i^{m_i}$ is a decomposition into a direct sum of simple H-modules, we have $\sum_i m_i^2 = 2$. Hence V decomposes into a direct sum of two simple H-modules. As H is a normal subgroup of order 2, it follows that both of these submodules must have the same dimension by a homework problem.

6. Let $\lambda = [3, 1, 1]$ and let A_5 be the subgroup of all $\pi \in S_5$ for which $\epsilon(\pi) = 1$.
(a) How does the simple S_5-module S^λ decompose as an A_5 module?
(b) Calculate the dimensions of all simple S_5-modules. (You should only use whatever has been PROVED in class and/or homeworks).
(c) Find a complete set of simple representations of A_5 up to isomorphism.

Solution: For (a), we use the criterion from the previous problem. So we calculate $\chi_{[3,1,1]}(\pi)$ for all $\pi \notin A_5$. It suffices to calculate this for $\pi = (12)$, $\pi = (123)(45)$ and $\pi = (1234)$ (the other cycle structures would be given by (123), (12)(34), (12345) and the identity, which are all in A_5). Using Murnaghan-Nakayama rule, it can be checked directly that indeed $\chi_{[3,1,1]}(\pi) = 0$ for $\pi \notin A_5$. Hence the representation decomposes into the direct sum of two simple A_5 modules of the same dimension. To prove (b), let d_λ be the dimension of S^λ. By using the restriction rule, we find that

$$d_{[3,1,1]} = d_{[2,1,1]} + d_{[3,1]} = d_{[1,1,1]} + 2d_{[2,1]} + d_{[3]} = 6.$$

Hence $S^{[3,1,1]}$ decomposes into the direct sum of two non-isomorphic simple three-dimensional A_5-modules. One checks similarly that $d_{[5]} = d_{[1^5]} = 1$, $d_{[4,1]} = d_{[2,1,1,1]} = 4$ and $d_{[3,2]} = d_{[2,2,1]} = 5$. For (c), one checks that $S^{[3,2]}$ and $S^{[4,1]}$ remain irreducible if restricted to A_5 as $\chi_{[3,2]}((12)) = 1$ and $\chi_{[4,1]}((12)) = 2$, again by using the Murnaghan-Nakayama rule. Hence we have two non-isomorphic simple 3-dimensional, one simple 4-dimensional and one simple 5-dimensional representation of A_5, besides the trivial representation. Adding up the squares of their dimensions, we obtain $60 = |A_5|$. Hence these are all the simple representations up to isomorphism, and

$$\mathbf{C}A_5 \cong \mathbf{C} \oplus M_3(\mathbf{C}) \oplus M_3(\mathbf{C}) \oplus M_4(\mathbf{C}) \oplus M_5(\mathbf{C}).$$