SOLUTION SKETCHES: MATH 202 QUAL EXAM SPRING 2009

The following contains outlines for the solutions. Please ask if you have further questions.

1. (a) By Frobenius' formula, we have

$$Tr_{V^{\otimes n}}(p_n d) = \sum_\lambda \chi_\lambda(p_n) s_\lambda = s_{[n]},$$

where s_λ indicates the Schur function in N variables $x_1, ..., x_N$, and the summation goes over all Young diagrams with $\leq N$ rows. Here we used the fact that p_n is a minimal idempotent in $(\mathbf{C}S_n)_{[n]}$ (see also the second problems in the ninth homework assignment of the winter quarter, and in the second homework assignment of the spring quarter).

(b) By definition of the action of d, we have

$$Tr_{V^{\otimes 10}}((p_4 \otimes p_4 \otimes p_2)d) = Tr_{V^{\otimes 4}}(p_4 d) Tr_{V^{\otimes 4}}(p_4 d) Tr_{V^{\otimes 2}}(p_2 d) = s_{[4]}^2 s_{[2]}.$$

Using the Pieri rule, we get $s_{[4]}s_{[2]} = s_{[6]} + s_{[5,1]} + s_{[4,2]}$. Again by using the Pieri rule, we see that $s_{[6]}s_{[4]}$ is a linear combination of Schur functions labeled by Young diagrams with at most two rows, which does not include $s_{[6,3,1]}$. On the other hand, one sees that we can obtain the diagram $[6, 3, 1]$ by adding four boxes in four different columns to both $[5, 1]$ and $[4, 2]$. Hence the correct answer is 2. *Alternatively* you can also get this result by calculating the Kostka number $K_{[4,4,2]}^{[6,3,1]}$.

(c) We have seen that the multiplicity of an S^n-module S^λ in $V^{\otimes n}$ is given by $s_\lambda(1, ..., 1)$, where we have N 1's, provided that λ has $\leq N$ rows. Hence the answer is 0 for $N = 3$ and for $N = 5$, using the results of a homework problem, it is equal to

$$\prod_{1\leq i<j\leq 5} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \frac{7 \cdot 6 \cdot 5 \cdot 2 \cdot 1}{4!} \; \frac{5 \cdot 4 \cdot 3}{3!} \; \frac{2!}{2!} = 175.$$

2. (a) It is easy to check that the degree must be $n(n-1)/2$. For (b), the key observation is that interchanging variables x_i and x_j results in interchanging columns i and j in the matrix $(\partial e_r/\partial x_j)$. Hence the determinant must be an antisymmetric function, and therefore divisible by $\Delta = \prod_{i<j}(x_i - x_j)$. Comparing degrees, we see that

$$(\partial e_r/\partial x_j) \; = \; \alpha\Delta$$

for a scalar α. In order to calculate the scalar, it suffices to calculate the leading coefficient in the determinant. Observe that the first row of the matrix only contains $1'$s, while the first column $(\partial e_r/\partial x_1)$ does not contain any x_1's. Hence the leading term must be in the $(1, 1)$ minor of the determinant expansion with respect to the first row. Moreover, observe that

$$e_r(x_1, ..., x_n) = x_1 e_{r-1}(x_2, ..., x_n) \; + \; e_r(x_2, ..., x_n).$$

Hence we can expand the $(1,1)$ minor as

$$\det(\partial e_r/\partial x_j)_{2\le r,j\le n} \;=\; x_1^{N-1}\det(\partial e_{r-1}/\partial x_j)_{2\le r,j\le n} \;+\; \textit{lower terms in } x_1,$$

where the elementary symmetric functions on the right hand side are in the variables x_2, ..., x_n. We can now show by induction on the number of variables (after checking the claim for $n=2$) that the leading term is equal to 1.

3. Part (a) was easy. For part (b), just observe that any eigenvalue λ of $\rho(g)$ must be a root of unity as G is a finite group. Hence $\lambda^{-1}=\bar{\lambda}$ and

$$Tr(\hat{\rho}(g)) = Tr(\rho(g^{-1})^t) = Tr(\rho(g^{-1})) = \bar{T}r(\rho(g)),$$

from which follows claim (b). To show (c), observe that we have by assumption

$$1 \le \langle \chi_0, \chi_V\chi_W\rangle \;=\; \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_W(g) \;=\; \langle \chi_V, \bar{\chi}_W\rangle.$$

By parts (a) and (b), we know that also $\bar{\chi}_W$ is a character, which also must be simple (just calculate $\langle\bar{\chi}_W,\bar{\chi}_W\rangle$). But then $\langle\chi_V,\bar{\chi}_W\rangle\ge 1$ only if $\chi_V=\bar{\chi}_W$.

4. Part (a) was basically a straightforward if tedious calculation. In order to calculate the intersection of the ideal with $k[x]$ however, one has to chose a monomial ordering with $y>x$. One calculates $f_3=y-x^2$ and $f_4=x^5-1$ and checks that these two poplynomials are a Gröbner basis for the ideal, using Buchbinder's algorithm. Similarly, it follows that f_4 generates its intersection with $k[x]$. From these two polynomials, one calculates the variety as consisting of the five points (θ,θ^2), with θ running through all fifth roots of unity.

5. For (a), one calculates the characters of all three nontrivial elements to be equal to -1, using Murnaghan-Nakayama. As the order of each of these elements is two, it follows that two of its eigenvalues have to be -1, and one has to be equal to 1. We obtain from this the Molien series

$$\Phi_G(z) = \frac{1}{(1-z)^3} \;+\; \frac{3}{(1-z)(1+z)^2} \;=\; \frac{1+z^3}{(1-z^2)^3}.$$

For (b), we obtain the same characters and hence also the same Molien series. Moreover, here it is easy to exhibit the three algebraically independent and G-invariant monomials $\theta_i=y_i^2$, $i=1,2,3$, and $\eta=y_1y_2y_3$. Comparing with the Molien series, we see that

$$k[y_1,y_2,y_3]^{\tilde{G}} = k[\theta_1,\theta_2,\theta_3]\oplus\eta k[\theta_1,\theta_2,\theta_3].$$

Moreover, we have the relation $\eta^2 = \theta_1\theta_2\theta_3$. As the action in (a) has the same character as the one in (b), their rings of invariants are isomorphic, and hence given by the same relations.