Here are a number of exercises which you should do. They are (not supposed to be) hard, but require you to know and understand basic concepts and definitions of the lecture.

1. This exercise probably gives the simplest example of a Banach space for which the second dual is strictly larger than the original space. We denote by \( c_0 \) the space of all sequences \( a = (a_n) \) of real numbers, with \( \lim_{n \to \infty} a_n = 0 \), and with \( \|a\| = \sup |a_n| \).
   (a) Show that any linear functional \( \phi \) which is defined everywhere on \( c_0 \) is uniquely determined by a sequence \( f = (f_n) \) such that \( \phi(a) = f \cdot a \); here \( \cdot \) denotes the usual dot product.
   (b) Show that a sequence \( f = (f_n) \) corresponds to a functional \( \phi \) in \( c_0^* \) if and only if \( f \in \ell^1 \), where \( \ell^1 \) consists of all sequences \( (f_n) \) with \( \sum |f_n| < \infty \).
   (c) Show that \( \|\phi\| = \|f\|_1 \). Here \( \|\phi\| \) is the usual norm on dual spaces, and \( \|f\|_1 = \sum |f_n| \).
   (d) Show that the dual of \( \ell^1 \) is given by the set \( \ell^\infty \) of all bounded sequences.

2. Let \( X \) be a topological vector space (TVS). Recall that a subset \( B \subset X \) is called bounded if for any open neighborhood \( U \) of 0 there exists an \( \varepsilon > 0 \) such that \( \varepsilon B \subset U \). Moreover, if \( X \) is a locally convex space, whose topology is defined by seminorms \( p_n \), then we call a sequence \( (x_n) \subset X \) to be a Cauchy sequence if for given \( \varepsilon > 0 \) and given \( p_n \) we can find an \( N \) such that \( p_n(x_m - x_n) < \varepsilon \) whenever \( n, m \geq N \). Show that any Cauchy sequence in \( X \) is bounded.

3. Let \( L \) be the functional on the space of test functions \( \mathcal{D}(\mathbb{R}) \) given by \( L(\phi) = \int_0^\infty \phi(x) \, dx \).
   (a) Check that \( L \) is a distribution, i.e. it is a well-defined linear functional on \( \mathcal{D}(\mathbb{R}) \) which is continuous with respect to the topology on \( \mathcal{D}(\mathbb{R}) \) as defined in the lecture. For showing continuity it suffices (see Prop. IV.5.21) that if we have a sequence \( (\phi_n) \) for which there exists a compact set \( K \) with \( \text{supp} \phi_n \subset K \) for all \( n \) such that \( \phi_n \to 0 \) uniformly on \( K \) for all \( j \), then also \( L(\phi_n) \to 0 \).
   (b) Determine all derivatives \( L^{(j)} \) of \( L \), where \( L^{(j)}(\phi) = (-1)^j L(\phi^{(j)}) \) for all test functions \( \phi \in \mathcal{D}(\mathbb{R}) \).

4. Let \( X = C(G) \) be the Banach space of continuous functions on a compact group, with supremum norm, and let \( X^* \) be its dual, the space of all Borel measures on \( G \). Let \( \ell_g(f)(h) = f(gh) \) for all \( g, h \in G \) and \( f \in C(G) \), which induces an action on \( C(G)^* \), also denoted by \( \ell_g \), by
   \[ \ell_g(\mu)(f) = \mu(\ell_g(f)). \]
   Show that the action of \( G \) on \( C(G)^* \) is equi-continuous with respect to the weak-* topology. This means that for any weak-* open neighborhood \( U \) of 0 (the zero element of \( C(G)^* \)), there exists a weak-* neighborhood \( V \) of 0 such that \( \ell_g(V) \subset U \) for all \( g \in G \).

5. (hard) Let \( K \subset X \) be a convex compact subset of the topological vector space \( X \), and let \( (T_g) \), \( g \in G \) be a group of linear maps \( T_g : X \to X \) such that \( T_g \circ T_h = T_{gh} \) for all \( g, h \in G \). Assume that \( (T_g) \) acts equi-continuous on \( X \) and that \( T_g(K) \subset K \) for all \( g \in G \). Show that there exists an \( x_0 \in K \) such that \( T_g(x_0) = x_0 \) for all \( g \in G \). (Hint) The idea is to find a minimal nonempty convex compact subset \( H \subset K \) for which \( T_g(H) \subset H \). The crucial point is to show if such an \( H \) contains two distinct points that there exists a smaller convex compact \( H_1 \subset H \) which only contains one of these points. One of the standard tricks is to consider the set \( H - H = \{ h_1 - h_2, h_1, h_2 \in H \} \). Let me know if you need more hints).

6. Show that any compact group \( G \) has a Haar measure.

7. (a) Let \( H \) be a Hilbert space and \( a \in B(H) \). Show that \( a \) is not invertible if there exists a sequence of vectors \( (v_n) \subset H \) with \( \|v_n\| = 1 \) for all \( n \) such that \( \|av_n\| \to 0 \) if \( n \to \infty \). (Hint: Show that the inverse of \( a \) can not have finite norm).
   (b) Let now \( H \) be a Hilbert space with orthonormal basis \( (\xi_n) \), \( n \in \mathbb{Z} \). Let \( u : H \to H \) be given by \( u \xi_n = \xi_{n+1} \). What is the spectrum \( \sigma(u) \) of \( u \)? (Hint: Consider the vectors \( \nu_n = \frac{1}{n} \sum_{i=1}^{n} \lambda^{-i} \xi_i \) for suitable \( \lambda \).
   (c) What is the spectrum of \( u + u^* \)? Give a description of the \( C^* \) algebras \( C^*(u) \) and \( C^*(u + u^*) \) as algebras of continuous functions on a suitable compact set.