MODULAR FORMS FACTS AND HOMEWORK VI

Let H_+ be the upper half plane of the complex plane, i.e. the set of all complex numbers $\tau = \alpha + i\beta \in \mathbf{C}$ for which $Im(\tau) = \beta > 0$. A differentiable function F on H_+ is called a *modular form* of rank k if

$$F(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{2k} F(\tau)$$

for any integers a, b, c, d satisfying $ad - bc = 1$. It can be shown that any modular form F can be written as a power series

$$F(\tau) = \sum_{n=-\infty}^{\infty} c_n q^n, \qquad \text{where } q = e^{2\pi i\tau}.$$

You should check for yourself that $Im(\tau) > 0$ implies $|q| < 1$.

Examples 1. Consider the function

$$G_{2k}(\tau) = \sum_{(n,m)\neq(0,0)} \frac{1}{(m\tau + n)^{2k}},$$

where the summation goes over all pairs of integers $(n, m) \neq (0, 0)$, and $k \geq 2$. It can be shown that these series do converge for all $\tau \in H_+$. It was shown in class that G_{2k} is a modular form of rank k. It is known that all modular forms of rank < 6 are of this form (up to a scalar multiple).

2. Another important function, defined on H_+, is given by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where we substituted $q = e^{2\pi i\tau}$. It can be shown that this infinite product does converge for any $\tau \in H_+$, i.e. for which its imaginary part is positive. It can also be shown that η^{24} is a modular form of rank 6 which is not equal to G_{12}.

Exercises

1. Let F be a modular form of rank k. Express $F(\tau + 1)$ and $F(-1/\tau)$ in terms of $F(\tau)$.
2. (optional). Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with integer coefficients and with $|det(A)| > 1$. Show that the map $\mathbf{x} \mapsto A\mathbf{x}$ maps the set $\mathbf{Z}^2$ of vectors with integer coefficients into $\mathbf{Z}^2$, but that it is not onto. *Hint* : Show first that A^{-1} has some coefficients which are not integers. Then find a vector $\mathbf{y} \in \mathbf{Z}^2$ such that $A^{-1}\mathbf{y}$ is NOT in $\mathbf{Z}^2$.
3. Do problem 43.4 (a) and (c) of the Friendly Introduction. I plan to do 43.4(b) in class. Here are the problems:

(a) Calculate $a_p = p - N_p$ for $y^2 = x^3 + p$.

(c) Same for $y^2 = x^3 - x^2 + p$. You should check in both cases that p is a bad prime, i.e. p divides the discriminant of the corresponding elliptic equations.

If you have spare time and energy, you could already start looking into Exercises 44.1 and 44.2 of the Friendly Introduction.