2.3 (5) It follows from the binomial theorem that

\[(1 + d)^n = \sum_{j=0}^{n} \binom{n}{j} d^j \geq 1 + nd,\]

as all the summands are positive and \(\binom{n}{1} = n\). It follows that

\[|c^n| = \frac{1}{(1+d)^n} \leq \frac{1}{1 + nd} \leq \frac{1}{nd}.\]

2.3 (6) We use the Comparison Lemma with \(b_n = c^n, b = 0, a_n = \frac{1}{n}\) and \(C = d\), where \(d\) is as in the previous exercise. We have already shown in class that \((a_n)\) converges to 0. Using the previous exercise, we obtain

\[|c^n - 0| = |b_n - 0| \leq d|\frac{1}{n} - 0| = d|a_n - 0|.\]

Hence the conditions of the Comparison Lemma are satisfied with \(b = 0\). This proves the claim.

To show part (b), recall that we have shown in Problem 2(a) on page 33 that the sequence \(\left(\frac{1}{\sqrt{n}}\right)\) converges to 0. We now can use the Comparison Lemma with \(a_n = \frac{1}{\sqrt{n}}\) and \(b_n = \sqrt{nc^n}\), using

\[|\sqrt{nc^n} - 0| \leq d|\frac{\sqrt{n}}{n}| = d|\frac{1}{\sqrt{n}} - 0|.\]

This shows the claim.

2.3 (9) This is similar to the proof of Theorem 2.29. As \([a_{n+1}, b_{n+1}] \subset [a_n, b_n]\), we have

\[a_n \leq a_{n+1} \leq b_{n+1} \leq b_n\]

for all \(n \in \mathbb{N}\). It follows that the sequence \((a_n)\) is monotonously increasing. As every \(a_n\) is in \([a_1, b_1]\), it also follows that \(a_n \leq b_1\) for all \(n\). Hence \((a_n)\) is also bounded above. It follows from the Monotone Convergence Theorem that the sequence \((a_n)\) converges to a number we call \(a\). Similarly, we show that \((b_n)\) is monotonously decreasing and bounded below by \(a_1\). Hence it converges to a number \(b\). Finally, as \(b_n - a_n \geq 0\), we also have

\[0 \leq \lim_{n \to \infty} (b_n - a_n) = b - a,\]
by Lemma 2.21. Hence \( a \leq b \). Moreover, by the MCT, we have
\[
a = \sup\{a_n, n \in \mathbb{N}\}.
\]
Hence \( a_n \leq a \) for all \( n \in \mathbb{N} \). One shows similarly that \( b_n \geq b \) for all \( n \in \mathbb{N} \).

2.4 (12) Let \( \ell \) be the positive solution of \( x^2 - x - c = 0 \), i.e. \( \ell = \frac{1}{2}(1 + \sqrt{1 + 4c}) \), and let
\[
f(x) = x^2 - x - c = 0.
\]
Observe that the graph of \( f \) is a parabola. Looking at the graph, we see that for positive \( x \) we have
\[
\begin{align*}
f(x) > 0 & \iff x^2 > x + c \iff x > \ell \\
f(x) < 0 & \iff x^2 < x + c \iff x < \ell
\end{align*}
\]
If \( x_n \) is positive, then also \( x_{n+1} = \sqrt{x_n + c} > 0 \). Hence, as \( x_1 > 0 \), it follows by induction that \( x_n > 0 \) for all \( n \). Moreover, it follows from the formula above that
\[
x_n^2 - x_{n+1}^2 = x_n^2 - (x_n + c) = f(x_n) > 0 \quad \text{for } x_n > \ell.
\]
Hence we obtain by induction that
\[
x_{n+1} < x_n \quad \text{for all } n \in \mathbb{N}, \quad \text{if } x_1 > \ell.
\]
It follows that \( (x_n) \) is a monotonously decreasing sequence bounded below by 0. Applying the Monotone Convergence Theorem, we conclude that \( (x_n) \) converges, i.e. there exists a number \( x \) such that
\[
\lim_{n \to \infty} x_n = x.
\]
Similarly as in Example 2.31, we use the convergence of subsequences, Proposition 2.30, to show that
\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} \sqrt{c + x_n} = \sqrt{c + x}.
\]
It follows that
\[
x^2 = x + c \iff f(x) = 0.
\]
Hence \( x \) is equal to the positive root of \( f(x) \), i.e. \( x = \ell \). We have shown that \( (x_n) \) converges to \( \ell \). The case with \( 0 < x_1 < \ell \) can be shown similarly, by proving that \( (x_n) \) is a monotonously increasing sequence bounded above by \( \ell \).