3.5 (3) Let $\epsilon > 0$. We estimate $|f(x) - f(x_0)|$ as follows, using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$: 

$$|f(x) - f(x_0)| = |(x - x_0)(x^2 + xx_0 + x_0^2)| \leq |x - x_0| (|x|^2 + |x||x_0| + |x_0|^2).$$

If $|x - x_0| < 1$, then we have $|x| < |x_0| + 1$ (check for the two different cases $|x_0| \geq 0$ and $|x_0| < 0$). Using this and the equation above, we obtain

$$|f(x) - f(x_0)| \leq |x - x_0| (|x_0| + 1).$$

Hence if

$$\delta = \frac{\epsilon}{(3|x_0|^2 + 3|x_0|) + 1},$$

and $|x - x_0| < \delta$, we obtain

$$|f(x) - f(x_0)| \leq |x - x_0| (3|x_0|^2 + 3|x_0| + 1) < \delta(3|x_0|^2 + 3|x_0| + 1) = \epsilon.$$

**Remark** Observe that our choice of $\delta$ did not just depend on $\epsilon$, but also on $x_0$.

3.5 (5) Let $\epsilon > 0$, and let $u, v \in \mathbb{R}$. Then we get the following estimate:

$$|h(u) - h(v)| = \left| \frac{(1 + v^2) - (1 + u^2)}{1 + u^2} \right| = \frac{|u - v| |u + v|}{|(1 + u^2)(1 + u^2)|}.$$

We have $0 \leq (1 - u)^2 = 1 + u^2 - 2u$ from which we conclude

$$2u \leq 1 + u^2 \quad \text{and} \quad \frac{|u|}{|(1 + u^2)|} \leq \frac{1}{2}.$$

We deduce from this and $1/(1 + u^2) \leq 1$ that

$$\frac{|u + v|}{|(1 + v^2)(1 + u^2)|} \leq \frac{|u|}{|(1 + u^2)|} \frac{1}{|(1 + u^2)|} + \frac{|v|}{|(1 + v^2)|} \frac{1}{|(1 + v^2)|} \leq 1.$$

Combining this with the first equation for this problem, we obtain

$$|h(u) - h(v)| \leq |u - v|.$$

Choosing $\delta = \epsilon$, it follows that $|h(u) - h(v)| < \epsilon$ whenever $|u - v| < \delta = \epsilon$. 

SOLUTIONS TO SELECTED HOMEWORK PROBLEMS II
3.5 (7) (a) Let $\epsilon > 0$. Let us first assume $x_0 > 0$. Then we have

$$|f(x) - f(x_0)| = \sqrt{x} - \sqrt{x_0} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}}.$$ 

Choose $\delta = \sqrt{x_0}\epsilon$. Then we have for $|x - x_0| < \delta$ that

$$|f(x) - f(x_0)| = \frac{|x - x_0|}{\sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \frac{\delta}{\sqrt{x_0}} = \epsilon.$$ 

If $x_0 = 0$, we obtain

$$|f(x) - f(0)| = \sqrt{x}.$$ 

If we choose $\delta = \epsilon^2$, we obtain $|f(x) - f(0)| < \epsilon$ whenever $|x - 0| < \epsilon$.

(b) By Theorem 3.17 any continuous function on a closed interval is uniformly continuous. Hence our $f$ in (a) is uniformly continuous on the interval $[0, 1]$.

(c) We have seen in part (a) that

$$|f(u) - f(v)| = \frac{1}{\sqrt{u} + \sqrt{v}} |u - v|.$$ 

If $C > 0$, we can always find values of $u$ and $v$ such that

$$\frac{1}{\sqrt{u} + \sqrt{v}} > C,$$

e.g. $u = 1/2C^2$ and $v = 0$. Hence $f$ is not a Lipschitz function.

3.5 (8) Observe that if we restrict the function $f$ to the interval $[0, 2p]$, then this restriction is uniformly continuous, by Theorem 3.17. This means that for every $\epsilon > 0$ we can find a $\delta_0 > 0$ such that

$$|u - v| < \delta_0 \Rightarrow |f(u) - f(v)| < \epsilon \quad u, v \in [0, 2p].$$ 

We claim that this is also true for all $u, v \in \mathbb{R}$ which satisfy $|u - v| < \delta = \min\{\delta_0, p\}$. Let us call the smaller number $v$, and let $m$ be the largest integer such that $mp \leq v$.

Then we have

$$0 \leq \tilde{v} = v - mp < p \quad \text{and} \quad 0 \leq \tilde{u} = u - mp < 2p,$$

where the second inequality follows from $v - mp = (v - u) + (u - mp) < 2p$. Hence we have

$$|\tilde{u} - \tilde{v}| = |u - v| < \delta \quad \text{and} \quad \tilde{u}, \tilde{v} \in [0, 2p].$$ 

By periodicity, $f(\tilde{u}) = f(u)$ and $f(\tilde{v}) = f(v)$. It follows

$$|f(u) - f(v)| = |f(\tilde{u}) - f(\tilde{v})| < \epsilon \quad \text{if} \quad |u - v| < \delta.$$ 

3.6 (2) (a) $f(x) = \frac{1}{x} - \frac{1}{1-x}$.

(b) $f(x) = \sin^2(2\pi x)$.

(c) $f(x) = \frac{x}{1+|x|}$.

3.6(x) The other homework problems listed will not be relevant for the midterm. But they will be due for the next homework assignment.