1. (a) Let \( X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). Show that \( \exp(tX) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \).

(b) Show that any \( g \in SO(2) \) can be written as \( \exp(tX) \) for some \( t \in \mathbb{R} \). This implies that the map \( \exp \) from the Lie algebra \( so_2 \) to the Lie group \( SO(2) \) is surjective.

(c) Show that \( O(2) \) has the same Lie algebra as \( SO(2) \). (Hint: consider the map \( t \in \mathbb{R} \mapsto \det(\exp(tX)) \) for any \( X \) in the Lie algebra of \( O(2) \).

(d) Find a Lie group \( H \) and two homomorphisms \( \Phi_i : O(2) \to H \), \( i = 1, 2 \) which have the same Lie algebra map but are not the same.

2. (Orthogonal diagonalization) It can be easily seen that the eigenvalues of \( \exp(tX) \) in the previous problem are equal to \( e^{\pm it} \). So in general we can not diagonalize orthogonal matrices over the real numbers. We will show here that we can still conjugate any orthogonal matrix to a matrix with a diagonal block with eigenvalues \( \pm 1 \) and \( 2 \times 2 \) diagonal blocks as in Problem 1.

(a) Let \( \lambda \) be an eigenvalue of the orthogonal matrix \( g \in O(n) \). Show that \( |\lambda| = 1 \).

(b) Let \( v \) be an eigenvector of \( g \) with eigenvalue \( \lambda \notin \mathbb{R} \). Then we can write \( v = v_1 + iv_2 \), with \( v_1, v_2 \in \mathbb{R}^n \). Show that \( (v, v) = 0 \) and \( (v_1, v_1) = (v_2, v_2) \).

(c) Assume \( \lambda = e^{it} \), \( \lambda \notin \mathbb{R} \). Show that the action of \( g \) on the span of \( v_1 \) and \( v_2 \) is given by the matrix \( \exp(tX) \) as in the first problem.

(d) Show that there exists an orthonormal basis \( \{u_1, u_2, \ldots, u_n\} \) of \( \mathbb{R}^n \) consisting of eigenvectors of \( g \) with eigenvalues \( \pm 1 \) or of pairs of vectors \( v_1 \) and \( v_2 \) as in (c) belonging to the eigenvalues \( e^{\pm it} \).

3. Show that \( SO(n) \) is contractible for all \( n \in \mathbb{N} \). (Hint: Use the last problem to show that for given \( g \in SO(n) \) there exists an orthogonal matrix \( u \) such that \( g = u d u^{-1} \), where \( d \) only consists of \( 2 \times 2 \) diagonal blocks as in Problem 1 or diagonal entries equal to \( \pm 1 \).

4. Show that the Lie algebra of \( U(n) \) is given by all matrices \( X \) such that \( X^* = -X \). What is the Lie algebra of \( SU(n) \)? What are the dimensions of these Lie algebras?

5. Do Problem 10 and 12 on page 60 of the course book.