Recall:

Theorem: If \( H \triangleleft G \) (normal subgroup),

- can construct factor group \( G/H \) as follows:
  \[
  G/H = \{ aH, \ a \in G \} \text{ left cosets of } H
  \]
- operation: \( (aH)(bH) = abH \)

Proof: already showed
- operation is well-defined
  (i.e. if \( \alpha H = \alpha' H \), \( bH = b' H \)
  \[
  \Rightarrow (\alpha' H)(b' H) = (\alpha H)(bH) \\
  \\
  \Rightarrow \alpha' b' H = \alpha b H
  \]
  (Remain: not true for non-normal subgroups !)
Associativity: \((aH \cdot bH) \cdot cH = abH \cdot cH = (ab)H \cdot cH\)

\[ aH \cdot (bH \cdot cH) = aH \cdot bcH = a((bc)H) \]

Identity element: \(eH = H\)

Inverse: \((aH)^{-1} = a^{-1}H\) (check for yourself!)

Examples:

\[ G = \{2, 4\} \quad H = \{4, 2\} \]

\[ G/H = \{ H, 1+H, 2+H, 3+H \}\]

\[ (2+H) + (3+H) = 5+H = 1+H \]

Remark: Can show:

\[ \phi: \frac{2}{4} \rightarrow 24, \quad i+42 \rightarrow i \]

defines an isomorphism!
Remark: $A_4$ is not abelian, but its factor group $A_4/4 \cong S_3$ is abelian.

More examples later.

Chapter 8: External Direct Products

Idea: Given 2 or more groups, build a bigger group.

Definition: Let $G_1, G_2, \ldots, G_n$ be groups. Then the external direct product $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ consists of all elements of the form $(g_1, g_2, g_3, \ldots, g_n)$ where $g_i \in G_i$, $i = 1, 2, \ldots, n$. 
Theorem: The external direct product $G_1 \oplus \cdots \oplus G_n$ forms a group under coordinate wise group operation i.e. $(g_1, \ldots, g_n) \cdot (h_1, \ldots, h_n) = (g_1h_1, g_2h_2, \ldots, g_nh_n)$

Proof: Exercise

E.g. if $e_i \in G_i$ is the identity element of $G_i$

$(e_1, e_2, \ldots, e_n)$ is $\ldots$ $e_1 \oplus e_2 \oplus \cdots \oplus e_n$

Examples: 

O  $\mathbb{U}(6) \oplus \mathbb{U}(8)$

$\mathbb{U}(6) = \{ 1, 5 \}$

$\mathbb{U}(8) = \{ 1, 3, 5, 7 \}$

$\Rightarrow \mathbb{U}(6) \oplus \mathbb{U}(8) = \{ (1,1), (1,3), (1,5), (1,7) \}$

Ex: $(5,5), (5,7) = (25,35) \pmod{6}, \pmod{8} = (1,3)$
\( \mathbb{Z}_2 \oplus \mathbb{Z}_3 = \{ (0,0), (0,1), (0,2), (1,0), (1,1), (1,2) \} \)

Additive notation:

**Question:** Is this a new group of order 6 up to isomorphism?

So far we know 2 groups of order 6, \( \mathbb{Z}_6 \) and \( S_3 \), non-isomorphic

Is \( \mathbb{Z}_2 \oplus \mathbb{Z}_3 \) isomorphic to \( \mathbb{Z}_6 \)?

\( \equiv \) Can we find an element of order 6 in \( \mathbb{Z}_2 \oplus \mathbb{Z}_3 \)?

Check order for each element.

Try: \( (1,1) \).

Additive notation:

\( 2(1,1) = (2,2) = (0,2) \) \( \uparrow \) \( \equiv \) \( \text{ord} \ (1,1) = 6 \)

\( 3(1,1) = (3,3) = (1,0) \) \( \uparrow \text{mod 2} \) \( \uparrow \text{mod 3} \)

Have excluded 1, 2, 3

\( \equiv \text{ord} \ (1,1) = 6 \).
Conclusion: \( \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6 \)

Isomorphism given by \( j \in \mathbb{Z}_6 \rightarrow j \cdot (1) = (j \cdot 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_3 \)

(again: \( \text{ord}(1,1) = 6 \Rightarrow \langle (1,1) \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \)

as \( |\mathbb{Z}_2 \oplus \mathbb{Z}_3| = 6 \).)

3. \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \)

\[ \{ (0,0), (1,0), (0,1), (1,1) \} \]

Check:

- All have order 2

\( \Rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is a group of order 4

but it does NOT contain an element of order 4

\( \Rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is NOT isomorphic to \( \mathbb{Z}_4 \)
Remark: One can show that $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are the only groups of order 4 up to isomorphism, i.e., any group $G$ with $|G| = 4$ must be isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Useful Theorem:

Theorem: Let $G$ be a group with $a^2 = e$ for all $a \in G$.

$\Rightarrow$ $G$ is abelian, i.e., $ab = ba$ for all $a, b \in G$.

Proof:

1. $a, b \in G 
\Rightarrow a^2 b^2 = e \cdot e = e$

2. $a^2 b^2 = e = (ab)^2 = ab \cdot ab \quad \text{left cancellation}
\Rightarrow ab = ba \quad \text{right cancellation}
ab = ba \quad \text{right cancellation}
2. \( G = \mathbb{Z}_{12}, \ H = \langle 3 \rangle \leq \mathbb{Z}_{12} \)

\[ H = \{0, 3, 6, 9\} \]

we have 3 cosets

\[ 1 + H = \{1, 4, 7, 10\} \]

\[ 2 + H = \{2, 5, 8, 11\} \]

Can show: \( G/H \cong \mathbb{Z}_3 \)

3. let \( H = \langle \text{id}, (21)(3)\rangle, (12)(3), (14)(23) \rangle \leq A_4 \)

have shown: \( H \trianglelefteq A_4 \)

can form factor group \( A_4/H \)

\[ |A_4/H| = 3 \Rightarrow A_4/H \cong \mathbb{Z}_3 \]

\( A_4/H = \langle \ H, (123) H, (132) H \rangle \)

\( (132) = (123)^2 \)

\( \text{check: } (32) \neq (123)H \)