Review for midterm:

Material: mostly Chapters 5, 6, 7

ch 5 permutation groups.

\[ S_n = \{ \text{all permutations of } 1, 2, \ldots, n \} \]

\[ |S_n| = n! \]

Main results:

- any permutation can be written as a product of disjoint cycles
- if disjoint cycles of perm. \( \pi \) have lengths \( m_1, m_2, \ldots, m_r \)

\[ \Rightarrow \text{ord}(\pi) = \text{e.c.m. } (m_1, m_2, \ldots, m_r) \]
Example: practice exam

\[ \pi = (1234)(14) \]

Calculate \( \pi^{1802} \)

So, calculate \( \text{ord}(\pi) \) first

\[ (1234)(14) = (1)(234) \]

\[ \Rightarrow \text{ord}(\pi) = 3 \]

\[ \Rightarrow \pi^{1802} \equiv 2 \pmod{3} \]

it follows \( \pi^{1802} = \pi^2 = (234)^2 = (243) \)
Problem 2 in practice exam:

how many elements of order 10 in $S_7$?

If $τ = C_1 C_2 \ldots C_r$ disjoint cycles, $\text{length of } C_i = m_i$:

$\text{ord}(τ) = \text{l.c.m.} (m_1, m_2, \ldots, m_r) = 10 \Rightarrow \begin{cases} m_1 + m_2 + \ldots + m_r = 10 \\ \text{l.c.m.} (m_1, m_2, \ldots, m_r) = 10 \end{cases}$

obviously in $S_7$ no 10-cycle

$\Rightarrow$ as $s | 10$ need a cycle of length $s$

only possibility: one 5-cycle
one 2-cycle

how many 5-cycles in $S_7$?

we have \( \binom{7}{5} = \binom{7}{2} = 21 \) subsets with 5 elements a $\{(1, 2, \ldots, 7)\}$

pick one of those subsets, say $\{(1, 2, 3, 4, 5)\}$

how many 5-cycles in $S_5$?
Solution: a 5-cycle in $S_5$ given by writing the numbers 1, 2, 3, 4, 5 in random order (e.g. (3 5 2 4 1)) have $5!$ possibilities.

Observe: applying a cyclic permutation to it we still get some permutation!

E.g. \((3 5 2 4 1) = (5 2 4 1 3) = (2 4 1 3 5)\)

\[\Rightarrow \text{ have } \frac{5!}{5} = 4! \text{ different 5-cycles in } S_5\]

We have seen: have \(\binom{5}{2} = 21\) subsets of \(\{1, 2, \ldots, 7\}\) with 5 elements.

\[\Rightarrow \text{ have altogether } 21 \cdot 4! = 21 \cdot 24 \text{ 5-cycles in } S_7\]
This is essentially the proof of the more general theorem: let $m \leq n$.

Then we have $\binom{n}{m} \cdot (m-1)!$ $m$-cycles in $S_n$.

E.g. $n=7, m=5$:

$$\binom{7}{5} \cdot 4! = 21 \cdot 24 = 504$$

$3$-cycles in $S_4$ (checked explicitly before)

Solution to problem:

have seen: $21 \cdot 24 = \frac{48}{24} = 504$ $5$-cycles in $S_7$.

after fixing $5$-cycle, only $2$ numbers left for $2$-cycle, only one possibility for $2$-cycle.

Solution: $504$
other results: odd/even permutations

subgroup $A_n$ of even permutations

$|A_n| = \frac{n!}{2}$ elements.

Chapter 6 Isomorphism:

An isom. $\Phi: G \rightarrow H$ is a 1-1 and onto map satisfying $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in G$.

Important problem:

Given two groups $G$ and $H$, decide whether they are isomorphic or not.
General Strategy:

to prove \( G \cong H \):
- construct an isom. \( \phi: G \rightarrow H \)
- have already shown:
  if \( G, H \) both cyclic with \( |G| = |H| \)
  \[ G \cong H \]
  if \( G = \langle a \rangle \), \( H = \langle b \rangle \)
  isom. given by \( a^i \rightarrow b^i \), \( i = 0, 1, \ldots, |G| - 1 \).

to prove \( G \not\cong H \) show that some a prop. of an isom. are violated
  e.g. \( |G| \neq |H| \) \[ G \not\cong H \]
- \( \text{ord } \phi(a) = \text{ord } a \)
  if we can find \( k \) s.t. \( k a \not\in G, \text{ord } a \not= k \text{ for all } k \) \[ G \not\cong H \]
example: \( H = \{ \text{id}, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \}\) is a subgroup of \( A_4 \subset S_4 \).

\( H \) has 3 elements of order 2.

\[ \Rightarrow H \neq \mathbb{Z}_4 \]

\( \mathbb{Z}_4 \) only has one element of order 4, namely 2.

In practice exam:

Is \( \mathbb{U}(5) \cong \mathbb{U}(10) \)?

\[ \mathbb{U}(5) = \{1, 2, 3, 4\} \]
\[ \mathbb{U}(10) = \{1, 3, 7, 9\} \]

\( \mathbb{U}(5) \) cyclic: \( \mathbb{U}(5) = \langle 2 \rangle \)

\( \mathbb{U}(10) \) cyclic: \( \mathbb{U}(10) = \langle 3 \rangle \)

\[ (\mathbb{U}(5))' = (\mathbb{U}(10))' \]

\[ (2^2 = 4, 2^3 = 8 = 3 \mod 5, 2^4 = 16) \]
\[ (3^2 = 9, 3^3 = 27 = 7 \mod 10) \]
\[ 3^4 = 81 = 1 \mod 10 \]
\begin{align*}
\Rightarrow \text{ both } \mathbb{U}(5) \text{ and } \mathbb{U}(10) \text{ are cyclic groups of order 4} \\
\Rightarrow \mathbb{U}(5) \cong \mathbb{U}(10) \\
\text{isom. given by } 2^j \in \mathbb{U}(5) \rightarrow 3^j \in \mathbb{U}(10) \\
\text{for } j = 0, 1, 2, 3 \\
\text{automorphisms} \\
\alpha : \mathbb{G} \rightarrow \mathbb{G} \text{ where } \alpha \text{ is an isom.} \\
\text{have seen: If } \mathbb{G} = \mathbb{Z}_n \\
\text{we have } \phi(n) = \# \{ j, 0 < j < n, \gcd(j,n) = 1 \} \\
\phi(n) \text{ automorphisms of } \mathbb{Z}_n \\
\text{If } j \in \mathbb{U}(n) \text{ can define autom. } \alpha_j : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \\
\quad (\phi(n) = |\mathbb{U}(n)|) \\
\end{align*}