Recall: \(G, H\) groups
\[ \Phi: G \rightarrow H \text{ is a homomorphism if } \Phi(ab) = \Phi(a) \Phi(b) \text{ for all } a, b \text{ in } G. \]

Question: Given 2 groups \(G, H\), how many homomorphisms are there, \(\Phi: G \rightarrow H\)?

Examples: Assume \(G\) is cyclic
have seen: in this case \(G \cong \mathbb{Z}\) (if infinite)
or \(G \cong \mathbb{Z}_n\) (if finite)

Crucial observation:
If \(G\) is cyclic with generating element \(a\)
i.e. \(G = \langle a \rangle \Rightarrow \)
any hom. \( \Phi: G \rightarrow H \)
is already uniquely determined if we know \( \Phi(1) \)
indeed any element of \( G \) is of the form \( a^j \)
\[ \Phi(a^j) = \Phi(a)^j = \text{determined by } \Phi(a) \]

nom. properties

Example: Determine all homomorphisms from \( G = \mathbb{Z} \)
into \( H \subset \text{Aut} \), \( H = \{ \text{id}, (12)(24), (13)(24), (14)(23) \} \)

Solution: Define for any \( h \in H \) the hom.
\[ \Phi_h: 1 \rightarrow h \]
\[ \Rightarrow \Phi_h(j) = \Phi_h(1+1+\ldots+1) = \Phi_h(1)^j = h^j \]
(use add. notation) j times
(use mulipl. not. for H)
$\Phi_n$ does define a hom.

If $h = \text{id.}$ $\implies \Phi_n(j) = \text{id}$ for all $j \in \mathbb{Z}$

If $\text{ord}(h) = 2$ $\implies \Phi_n(j) = \begin{cases} e & \text{if } j \text{ is even} \\ h & \text{if } j \text{ is odd} \end{cases}$

$\implies \text{ker of } \Phi_n = \begin{cases} \mathbb{Z} & \text{if } h = e \\ \langle 2 \rangle & \text{if } h \neq e \end{cases}$

$\Phi_n(\mathbb{Z}) = \langle h \rangle$

**General Fact.** There exists exactly one hom. $\Phi_n : \mathbb{Z} \rightarrow H$ for every elem. $h \in H$.

and these are all possible group hom. from $\mathbb{Z}$ to the group $H$.
(2) Find all hom. from \( \mathbb{Z}_3 \) into \( H \), 
\( H \subset A_4 \) as before.

Solution: \( \mathbb{Z}_3 \) is cyclic \( \Rightarrow \) any hom. \( \Phi \) already completely determined by \( \Phi(1) \) 

Question: What can we take for \( \Phi(1) \)?

Try \( \Phi(1) = \text{id} \). \( \Rightarrow \Phi(j) = \text{id}^j = \text{id} \) for all \( j \in \mathbb{Z} \)

\( \Rightarrow \Phi(j + k) = \text{id} = \text{id} \circ \text{id} = \Phi(j) \Phi(k) \)

Try \( \Phi(1) = h \neq \text{id} \)

Say \( h = (12)(34) \)

Does this define a homomorphism?

\( \Rightarrow \Phi(2) = h^2 = \text{id} \)
\( \Phi(3) = h^3 = h \)

\( 3 \mod 3 = 0 \)
\( \Rightarrow \Phi(3) = \Phi(0) = \text{id} \)

\( \uparrow \)
\[ \Rightarrow \text{ no hom. } \Phi : \mathbb{Z}_3 \rightarrow H \text{ possible} \]

with \( \Phi(1) = h \neq \text{id} \)

Remark: Nonexistence of such hom. can also be seen from the fact \( \text{ord } \Phi(g) \nmid \text{ord}(g) \) for any \( g \in G \)

in our example:

\[ \text{ord}(1) = 3 \text{ in } \mathbb{Z}_3 \]

\[ \text{ord}(h) = 2 \text{ for } h \in H, \ h \neq \text{id} \]

\[ \Rightarrow \text{ no hom. } \Phi \text{ which would map 1 to } h. \]

Essentially we have proved the following theorem:
Theorem: Let $H$ be any group, $G = \langle a \rangle$ cyclic.

(a) If $\text{ord}(a) = \infty$ (i.e. $G \cong \mathbb{Z}$)

\[ \Rightarrow \text{there exists a hom } \Phi_n : \alpha \rightarrow h \]

for any $h \in H$.

These are all possible homomorphisms $G \rightarrow H$.

(b) If $\text{ord}(a) = n$

\[ \Rightarrow \Phi_n : \alpha \rightarrow h \text{ defines a homomorphism } \]

if and only if $\text{ord}(h) | n$.

Example: Find all homomorphisms $\Phi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_9$.

SOL: by theorem, $\Phi(1) = j$ defines a homomorphism.

\[ \Rightarrow \text{ord}(j) | \text{ord}(1) = 6 \]

$\mathbb{Z}_6$
Recall: \( j \in \mathbb{Z}_9 \implies \text{ord}(j) = \frac{9}{\text{gcd}(j,9)} = \begin{cases} 1 & j = 0 \\ 3 & j = 3, 6 \\ 9 & \text{otherwise}. \end{cases} \)

\[ \implies \text{ord}(j) \mid 6 \iff j \in \{0, 3, 6\} \]

\( \implies \) have exactly 3 homomorphisms \( \Phi: \mathbb{Z}_6 \to \mathbb{Z}_6 \):

- \( \Phi(k) = 0 \) for all \( k \in \mathbb{Z}_6 \) \( j = 0 \)
- \( \Phi(k) = 3k \mod 9 \) \( j = 3 \)
- \( \Phi(k) = 6k \mod 9 \) \( j = 6 \)
Fundamental Theorem of Finite Abelian Groups

Main result:
as stated in book: \( G \cong \text{direct product of cyclic groups} \)
\[ \uparrow \]
\( \text{arbitrary finite abelian group} \)

Alternative statement (better for explicit calculations)

1. If \( |G| = p^a \Rightarrow G \cong \mathbb{Z}_{p^{a_1}} \oplus \mathbb{Z}_{p^{a_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{a_r}} \)
   where \( a_1 + a_2 + \cdots + a_r = a \)

   Def. \((a_1, a_2, \ldots, a_r)\) is a partition of \( a \)
   if \( a_1 \geq a_2 \geq \cdots \geq a_r \geq 0 \) integers
   and \( a_1 + a_2 + \cdots + a_r = a \)

   \( \Rightarrow \) If \( \text{Par}(a) = \# \text{partitions of } a \)
   then there are exactly \( \text{Par}(a) \) nonisom. groups \( G \) of order \( p^a \)
2. If \( |G| = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s} \),

\[ G = G_{p_1} \oplus G_{p_2} \oplus \ldots \oplus G_{p_s} \]

where

\[ |G_{p_i}| = p^{a(i)} \]

and there are exactly

\[ \text{Par}(a_i) \cdot \text{Par}(a_2) \cdot \ldots \cdot \text{Par}(a_s) \]

nonisom. groups \( G \) with \( |G| = p_1^{a_1} \ldots p_s^{a_s} \).

Example: how many nonisom. abelian groups of order 100?

Solution:

\[ 100 = 2^2 \cdot 5^2 \]

\[ G = G_2 \oplus G_5 \]

\[ |G_2| = 2^2 \]

\[ |G_5| = 5^2 \]

2 possibilities:

- \( D_4 \) or \( Z_2 \otimes Z_2 \)
- \( Z_{25} \otimes Z_5 \otimes Z_5 \)

2 possibilities.
have $\text{Par}(2)$. $\text{Par}(2) = 2 \cdot 2 = 4$ possibilities.