Main topic for mid-term: differentiation

Def. \( f: S \rightarrow \mathbb{R} \) (\( S \) a subset of \( \mathbb{R} \))

\( f \) differentiable at \( a \in S \) (interior point of \( S \))

if \( \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \) exists

E.g., in practice exam:

First T/F question: \( f \) differentiable at 0

\( \frac{2}{9} \lim_{x \to 0} \frac{f(x^2)-f(0)}{x^2} = 0 \)

\( \lim_{x \to 0} \frac{f(x^2)-f(0)}{x^2} = f'(0) \cdot 0 = 0 \)
another practice exam problem:

Let $f(x) = \begin{cases} 
  x^2 & x \text{ rational} \\
  x^3 & x \text{ irrational} 
\end{cases}$

Show: $f$ differentiable at 0

Solution: need to show that

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} \quad \text{exists!}$$

$$\frac{f(x)}{x} = \begin{cases} 
  x & x \text{ rational} \\
  x^2 & x \text{ irrational} 
\end{cases}$$

in either case we have

$$\lim_{x \to 0} \frac{f(x)}{x} = 0$$
Practice exam problem 4(c): \( g \) differentiable on \( \mathbb{R} \)
Assume \( |g(x) - g(y)| \geq |x - y| \)
for every \( x, y \) in \( \mathbb{R} \)
prove that \( |g'(x)| \geq 1 \)
for every \( x \)

**Solution:**
\[
|g'(x)| = \lim_{y \to x} \left| \frac{g(y) - g(x)}{y - x} \right| = \lim_{y \to x} \frac{|g(y) - g(x)|}{|y - x|} \geq \lim_{y \to x} \frac{|y - x|}{|y - x|} = 1
\]
L'Hospital's Rule

Let \( s \in \mathbb{R} \cup \{\pm \infty\} \), e.g. a differentiable function (at least near \( s \)).

Assume \( \lim_{x \to s} \frac{f'(x)}{g'(x)} = L \) exists

and either \( \lim_{x \to s} f(x) = 0 = \lim_{x \to s} g(x) \) or \( \lim_{x \to s} f(x) = \infty = \lim_{x \to s} g(x) \)

\[ \lim_{x \to s} \frac{f(x)}{g(x)} = L \]
Practica exam 3(b).

\[
\lim_{x \to 0} \sqrt{x} \sin \left( \frac{1}{x} \right) = \lim_{x \to 0} \frac{\sin \left( \frac{1}{x} \right)}{\sqrt{x}}
\]

Check: \( \sin \left( \frac{1}{x} \right) \to 0 \) for \( x \to 0 \)

\( \sqrt{x} \to 0 \) for \( x \to 0 \)

Differentiable for \( x \neq 0 \)

\[
\Rightarrow \lim_{x \to 0} \frac{\sin (1/x)}{\sqrt{x}} = \lim_{x \to 0} \frac{\cos (1/x) \left( -\frac{1}{x^2} \right)}{1 + \frac{1}{2} x^{-3/2} - x^{-1}}
\]

\[
= \lim_{x \to 0} \frac{\cos \frac{1}{x}}{\frac{1}{2} x^{-1/2}}
\]
aside: \[ \lim_{x \to \infty} \left| \frac{\cos \frac{1}{x}}{\frac{1}{2}x^{1/2}} \right| \leq \lim_{x \to \infty} \frac{\sqrt{2}}{x^{1/2}} = 0 \]

\[ \Rightarrow \lim_{x \to \infty} \frac{\cos \frac{1}{x}}{\frac{1}{2}x^{1/2}} = 0 \]

mean value theorem for derivatives:

\[ f: (a,b) \to \mathbb{R} \text{ differentiable, cont'n. at } c \in (a,b) \]

\[ \Rightarrow \exists \ y \in (a,b) \text{ s.t. } \quad f'(y) = \frac{f(b) - f(a)}{b-a} \]
Taylor's Theorem

Assume \( f \) is \( n \) times differentiable in \((a, b)\),
\( c \in (a, b) \), \( \alpha \in (a, b) \)

\[
\Rightarrow \exists \, \beta \text{ between } \alpha \text{ and } c \text{ such that }
\]

\[
f(\beta) = \frac{f^{(n)}(c)}{n!} (\beta - c)^n + R_n(\beta)
\]

where \( R_n(\beta) = \frac{f^{(n)}(c)}{n!} (\beta - c)^n \)

Crucial Point: To show that \( f(\beta) \) is given by its Taylor series, we need to show that

\[
\lim_{n \to \infty} R_n(\beta) = 0
\]
Remarks

1. We have seen that there are \( \infty \)-times differentiable functions \( f \) such that the Taylor series does not converge to \( f(x) \) except for \( x = c \).

\[
  f(x) = \begin{cases} 
    e^{-1/x^2} & x \neq 0 \\
    0 & x = 0 
  \end{cases}
\]

(Here \( c = 0 \))

2. Convergence can be shown if we have some estimates about the size of the derivatives.
Example: Practice problems 5(c):

Given: \[ |f^{(n)}(x)| \leq n! \quad \text{for all } x \in (-1, 1) \]

Prove: \[ f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \quad \text{for all } x \in (-1, 1). \]

Solution: \( \triangle \) not enough to prove that series converges to \( f(x) \); need to show that it converges to \( f(x) \). \( \Longleftrightarrow \) need to show: \[ \lim_{n \to \infty} R_n(x) = 0 \]

By Taylor's theorem: \( \exists \gamma \) between 0 and \( x \) s.t.
\[ |R_n(x)| = \left| \frac{f^{(n)}(\gamma)}{n!} (x-0)^n \right| \leq \left| \frac{n!}{n!} x^n \right| \leq |x|^n \to 0 \quad \text{for } x \in (-1, 1) \]
claim.

(Denial: only works for \( |x| < 1 \))

Last class: Also showed how Taylor's theorem can be used to estimate values of a function.

we showed: \( 1 + \frac{3}{2}x \leq (1 + x)^{3/2} \)

Similarly can also be shown: \( (1 + x)^{3/2} \leq 1 + \frac{3}{2}x + \frac{3}{8}x^2 \)

Practice Problem 4

T/F question 4

Answer: \( F \)

e.g. \( f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} \)
we showed: all derivatives exist 
(last class) \[ f^{(n)}(0) = 0 \quad \forall n \]

\[ \Rightarrow \text{ Taylor series: } \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 \]

which \( f(x) = e^{-1/2}x^2 \neq 0 \)

for \( x \neq 0 \)

here: Taylor series identically zero

\( f(x) \neq 0 \) for \( x \neq 0 \)

f identical zero means \( f(x) \geq 0 \)

for all \( x \).