Main theorem last class:

For a power series \( \sum a_n x^n \)
there exists \( R \geq 0 \) (possibly \( R = \infty \))
\( \Rightarrow \sum a_n x^n \) converges if \( |x| < R \)
\( \Rightarrow \sum a_n x^n \) diverges if \( |x| > R \).

Essentially same result holds for power series of the form
\( \sum a_n (x-x_0)^n \) \( x_0 \) fixed number

Example: \( \sum_{n=0}^{\infty} \frac{1}{2^n} (x-3)^n \)

check: radius of convergence \( R = 2 \)

formal way: let \( y = x-3 \) \( \Rightarrow \) get power series \( \sum \frac{1}{2^n} y^n \)
Use theorem from last class

\[ \text{Series } \sum_{n=0}^{\infty} \frac{1}{2^n} y^n \text{ converges for } |y| < 2 \]

\[ \Rightarrow \text{ original series converges for } |x-3| < 2 \]

\[ \Rightarrow -2 < x-3 < 2 \]

\[ \Rightarrow 1 < x < 5 \]

---

Can define function

\[ f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < R \]

\[ f(x) \text{ is the limit of the partial sums } \sum_{n=0}^{k} a_n x^n \]

Are polynomials continuous? \( \rightarrow \) is \( f \) also continuous?
Warning

A limit of continuous functions may not be continuous.

Example:

Let \( f_n : [0,1] \to \mathbb{R} \)

\( f_n(x) = x^n \quad \text{continuous} \)

Check:

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = \begin{cases} 
0 & \text{if } x < 1 \\
1 & \text{if } x = 1
\end{cases}
\]

But the function \( f(x) = \begin{cases} 
0 & x < 1 \\
1 & x = 1
\end{cases} \)

is not continuous at 1.
Ch. 24 Uniform Convergence

There are different notions of convergence for functions!

Def 1 pointwise convergence:

Let \( S \subseteq \mathbb{R} \) and \( f_n: S \to \mathbb{R} \) functions. A function \( f: S \to \mathbb{R} \) is a pointwise limit of \( (f_n) \) if we say that \( (f_n) \) converges pointwise to \( f \).

(Notation \( f_n \to f \) pointwise)

If \( \lim_\limits_{n \to \infty} f_n(x) = f(x) \) for every \( x \in S \).
have seen: \( f_n(x) = x^n \) converges pointwise

to \( f(x) = \begin{cases} \ 0 & x < 1 \\ \ 1 & x = 1 \end{cases} \)

here \( S = [0,1] \).

**Def. 2** uniform convergence.

For \( f_n, f : S \rightarrow \mathbb{R} \) functions

we say the functions \( f_n \) converge uniformly to \( f \) if for every \( \varepsilon > 0 \) we can find an \( N \in \mathbb{N} \) such that \( |f_n(x) - f(x)| < \varepsilon \) for all \( n \geq N \) and for all \( x \in S \).

Remark: The point here is that \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \in S \).
Examples

1. Let \( f_n(x) = \frac{1}{n} \sin nx \)

   we see \( |f_n(x)| = \left| \frac{1}{n} \sin nx \right| \leq \frac{1}{n} \)

   \( \Rightarrow \) if \( \varepsilon > 0 \), pick an integer \( N > \frac{1}{\varepsilon} \)

   \( \Rightarrow \) if \( m \geq N \) then

   \[ |f_n(x) - 0| = \left| \frac{1}{n} \sin nx \right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon \]

   for all \( x \) in \( \mathbb{R} \)

Result: \( f_n \) converges uniformly to the zero function \( f \)

i.e. \( f(x) = 0 \) for all \( x \).
Illustration of uniform convergence.

If \( |f_n(x) - f(x)| < \varepsilon \) for all \( x \),

\[ \Rightarrow \] the graph of \( f_n \)
has to be between the functions \( f(x) - \varepsilon \) and \( f(x) + \varepsilon \),
i.e., it has to be in the green region.

⚠️ If \( f_n \rightarrow f \) pointwise,
it may not necessarily converge uniformly.
Example 2 \( f_n(x) = x^n \quad \text{for} \quad S = [0,1] \)

We have seen \( f_n(x) \rightarrow f \) pointwise

\[
\begin{align*}
  f(x) &= 0 \quad x < 1 \\
  f(x) &= 1 \quad x = 1
\end{align*}
\]

\[ y = f(x) \]

\[ y = x^n \]

Pick \( \epsilon = 0.2 \)

In order to show \( f_n \rightarrow f \) uniformly
we would need to find an \( f_n \)
whose graph is in the green region.

\( f_n(x) = x^n \)

Picture suggests: whatever \( n \) we pick
the graph of \( f_n \) will not be in green region.
formal proof
want to show $f_n \nrightarrow f$ uniformly
i.e. need to find $\epsilon > 0$
such that for every $n \in \mathbb{N}$ we can find $x$ with

$$|f(x) - f_n(x)| > \epsilon$$

leq $\epsilon = 2$

however $f_n(x) = x^n$ is a cont. function

$f_n(0) = 0 \quad f_n(1) = 1$

$\Rightarrow \exists x \text{ s.t. } f(x) = \frac{1}{2} \quad (i.e. \ x = \frac{\sqrt[n]{1/2}}{n})$

$x \neq 1$ obvious $\Rightarrow f(x) > 0$

$\Rightarrow |f_n(x) - f(x)| = |\frac{1}{2} - 0| = \frac{1}{2} > 2$

$\Rightarrow$ no uniform convergence
have seen: uniform convergence is stronger than pointwise convergence.

Pointwise limit of contin. functions may not be continuous.

Next time we'll show:

uniform limit of contin. functions is again continuous.