General Remarks

midterm open book
unlikely question like the ones under 2

True/False questions on practice exam.

1. \( \sum \frac{x^{3m}}{\sqrt{n}} \) interval of convergence

radius of convergence:

\( \limsup \frac{1}{\sqrt{n}} \left( \frac{1}{3m} \right) \)

\( = \limsup n^{-\frac{1}{6m}} \)

radius of converg. = 1

does converge at \( x = -1 \)
by alternate series test.

Use ratio test for exponent \( \frac{e^{m}x}{e^{m/6}} \sim (e^{x}) \)
\( f_n \to f \) uniformly on \( C_0(1) \), \( f_n \) cauchy.

Is \( f \) bounded?

\( \square \)

(reason: each \( f_n \) is bounded (cauchy function on compact set!)

\( f_n \) cauchy sequence.

\( \Rightarrow \) uniformly bounded.

for large enough \( N \)

\[ |f_n(x) - f_{n'}(x)| < \epsilon \quad \forall x \in C_0(1) \]
3. \( f_n \to f \) pointwise, \( f_n \) cont., is \( f \) cont.?

\[ f_n(x) = x^n \]

\[ \lim_{n \to \infty} f_n(x) = f(x) \]

\[ f(x) \text{ is not cont.} \]

4. \( \sum 2^k a_n \) converges.

\[ \sum a_n x^k \text{ converges uniformly on } [0, 1) ? \]

Reason: Know for any power series \( \sum a_n x^n \)

- converges for \( |x| < R \)
- diverges for \( |x| > R \)

\[ \Rightarrow \text{for our series } R = 2 \]

\( \sum 2^k a_n \) would converge uniformly on any \([ -R, R ]\), \( R < 2 \) if \( R < 2 \) and converge.

Take \( R_1 = 1 \).
Does $\sum a_n x^n$ converge uniformly on $[-1,1]$?

Reason tricky:

By assumption, $\sum a_n x^n$ converges for

$x = 1$: $\sum a_{2n} + \sum a_{2n+1}$

and for $x = -1$: $\sum a_{2n} - \sum a_{2n+1}$

$\Rightarrow$ radius of convergence $R \geq 1$

Uniform if $R > 1$ $\checkmark$

If $R = 1$: it follows from Abel's Theorem.
\[ \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} \]

Claim: converges uniformly on \([-M, M] \quad \forall M > 0 \]

Proof: enough to show:
radius of convergence \( R = \infty \)

\[ \frac{1}{R} = \beta = \limsup_{n \to \infty} \left| \frac{\frac{1}{2^n} n!}{\frac{1}{2^{n+1}}} \right| = 0 \]

\[ \limsup_{n \to \infty} \left| \frac{1}{2^n} \frac{n!}{2^{n+1}} \right| = \frac{1}{2^n} n! \]

Case trick \( y = x^2 \)

Can use ratio test.
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{2^n n!}{2^{n+1} (n+1)!} = \lim_{n \to \infty} \frac{1}{2(n+1)} = 0 \]

\[ \Rightarrow \text{converges for all } y = )} \]
\[ = \sum_{n=0}^{\infty} \frac{\frac{x^{2n}}{2^n n!}}{2^n m!} \text{ converges for all } x. \]

(b) \text{ Claim: does } \textit{not} \text{ converge uniformly?}

\text{ Consider partial sum } \sum_{n=0}^{k} x^{2n} = \frac{x}{1 - \frac{x^2}{2}} \quad (S_n(x)) \text{ not uniformly Cauchy.}

\text{ i.e. for } \varepsilon = 1 \text{ and for all } N > 0 \text{ exists } \epsilon > N \text{ and } x \in \mathbb{R} \quad \forall |S_n(x) - S_m(x)| > 1

Observation: \[ |S_{k+1}(x) - S_k(x)| = \left| \frac{x^{2(k+1)}}{2^{k+1} (k+1)!} \right| \rightarrow \infty \quad \text{if } x \rightarrow \infty. \]
\[ \Rightarrow \quad \text{for } x \text{ large enough} \]

\[ |S_{n+1}(x) - S_n(x)| > 1 \]

for any \( k > N \).

5(c):

\[ f'(x) = \sum_{n=1}^{\infty} \frac{2m}{2^n n!} \times 2^{m-1} \]

\[ = \sum_{n=0}^{\infty} \frac{x^{2m}}{2^n n!} \]

\[ = x \sum_{n=1}^{\infty} \frac{1}{2^{n-1} (n-1)!} \times 2^{n-2} = 2(\sum_{n=1}^{\infty} \frac{1}{(n-1)!} ) \]

\[ \text{sub } n-1 \to n \]