 Name: ____________________

(6 points) 4. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in $\mathbb{R}^4$, with the following properties: $\{\mathbf{u}, \mathbf{v}\}$ are linearly independent, and $\mathbf{w}$ is *not* in the span generated by $\mathbf{u}$ and $\mathbf{v}$.

(a) Explain why $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are, in fact, linearly independent. [Hint: It might be easier to explain why it is impossible for the vectors to be linearly *dependent*.]

We have to show that the vector equation

$$x_1\vec{u} + x_2\vec{v} + x_3\vec{w} = \vec{0}$$

only has the trivial solution $x_1=0$, $x_2=0$ and $x_3=0$

~~you could also use Theorem 7 in section 1.7~~

If $x_3 \neq 0$ then $x_3\vec{w} = -x_1\vec{u} - x_2\vec{v}$

$$\vec{w} = -\frac{x_1}{x_3}\vec{u} - \frac{x_2}{x_3}\vec{v}$$

This means $\vec{w}$ would be in span $\{\vec{v}, \vec{w}\}$ contradicting our assumption.

Hence $x_3=0$ and we have

$$x_1\vec{u} + x_2\vec{v} + \underbrace{0}_{x_3\vec{w}} = \vec{0}$$

But as $\vec{u}$ and $\vec{v}$ are linearly independent, also $x_1=0$ and $x_2=0$

Hence $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent.

(b) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation satisfying

$$T(\mathbf{u}) + T(\mathbf{v}) = T(2\mathbf{u} + \mathbf{w}).$$

Is T one-to-one? Justify your answer.

As T is linear, ~~the~~ $T(\vec{u}) + T(\vec{v}) = T(\vec{u}+\vec{v})$

$\Rightarrow T(\vec{u}+\vec{v}) = T(2\vec{u}+\vec{w})$

$\Rightarrow T(\vec{u}+\vec{v}) - T(2\vec{u}+\vec{w}) = \vec{0}$

$\Rightarrow T(-\vec{u}+\vec{v}-\vec{w}) = \vec{0}$

~~But~~ $T(\vec{0}) = \vec{0}$

If ~~the~~ T is 1-1 $\vec{0} = -\vec{u}+\vec{v}-\vec{w}$ ($T(\vec{x})=\vec{0}$ only has solution $\vec{x}=0$)

This would contradict that $\vec{u}, \vec{v}$ and $\vec{w}$ are linearly independent.

$\Rightarrow$ $\boxed{\text{NO } T \text{ is not 1-1}}$