(6 points) 4. Let \( u, v, w \) be three vectors in \( \mathbb{R}^4 \), with the following properties: \( \{u, v\} \) are linearly independent, and \( w \) is not in the span generated by \( u \) and \( v \).

(a) Explain why \( \{u, v, w\} \) are, in fact, linearly independent. [Hint: It might be easier to explain why it is impossible for the vectors to be linearly dependent.]

We have to show that the vector equation
\[
x_1 \mathbf{u} + x_2 \mathbf{v} + x_3 \mathbf{w} = \mathbf{0}
\]
only has the trivial solution \( x_1 = 0, x_2 = 0 \) and \( x_3 = 0 \).

If \( x_3 \neq 0 \) then
\[
x_3 \mathbf{w} = -x_1 \mathbf{u} - x_2 \mathbf{v}
\]
This means \( \mathbf{w} \) would be in span \( \langle \mathbf{u}, \mathbf{v} \rangle \), contradicting our assumption.

Hence \( x_3 = 0 \) and we have
\[
x_1 \mathbf{u} + x_2 \mathbf{v} + 0 = \mathbf{0}
\]

But as \( \mathbf{u} \) and \( \mathbf{v} \) are linearly independent, also \( x_1 = 0 \).

Hence \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) are linearly independent.

(b) Let \( T: \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) be a linear transformation satisfying
\[
T(u) + T(v) = T(2u + w).
\]

Is \( T \) one-to-one? Justify your answer.

As \( T \) is linear,
\[
T(u + v) = T(u) + T(v) = T(u + v)
\]
\[
T(2u + w) = T(2u) + T(w) = T(2u) + T(w)
\]
\[
T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = T(\mathbf{u} + \mathbf{v} - \mathbf{w}) = \mathbf{0}
\]
\[
T(\mathbf{0}) = \mathbf{0}
\]

If \( T \) is \( 1 \) then
\[
\mathbf{0} = -\mathbf{u} + \mathbf{v} - \mathbf{w}
\]
This would contradict that \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) are linearly independent.

\[
\Rightarrow \text{NO, \ T is not 1-1}
\]