Math 18C NAME: PID: Section:

1. You have 50 minutes. **No** calculators, phones, books and notes allowed, except for one cheat sheet.
2. Write your solutions in the provided spaces. Show your work and justify your answers.

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 5 \\ 3 & 6 & 8 & 7 \end{bmatrix}$.

(a) Compute a basis for its column space.

Row Echelon Form $\begin{bmatrix} \boxed{1} & 2 & 3 & 2 \\ 0 & 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

1st and 3rd columns are pivot columns

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix} \right\}$ basis for column space

(b) Compute a basis for its null space.

We get the following equations from the echelon form:

$-x_3 + x_4 = 0 \Rightarrow \boxed{x_3 = x_4}$

$x_1 + 2x_2 + \underset{x_4}{\underset{\shortparallel}{3x_3}} + \cancel{x_4} = 0 \Rightarrow \boxed{x_1 = -2x_2 - 4x_4}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}}_{\text{basis}}$

(c) What is its rank?

rank = # pivot columns = 2

2. In each of the following examples, a vector space V is given with a subset $S \subset V$. Determine whether S is a subspace or not. In each case, explain why it is or is not a subspace.

(a) Let V be the vector space of all 2×2 matrices, and let S be the subset of all 2×2 matrices satisfying $A^T = A$.

Let A, B be in S

$\Rightarrow$ $(A+B)^T = A^T + B^T = A + B$ $\Rightarrow$ $A+B$ is in S

$(cA)^T = cA^T = cA$ $\Rightarrow$ cA is in S

$\Rightarrow$ S satisfies the conditions for a subspace.

(b) Let $V = \mathbf{P}_3$, the vector space of all polynomials of degree ≤ 3, and let $S = \text{span}\ \{x^2 + 1, 2\}$.

By theorem in class any span of vectors is a subspace

(c) Let $V = \mathbf{R}^2$, and let $S = \{\begin{bmatrix} s \\ t \end{bmatrix}, s \leq t\}$.

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in S but $(-1)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ is not in S

as $-1 > -2$

NOT a subspace

(d) Let V be the vector space of all 2×2 matrices, and let S be the subset of all invertible 2×2 matrices.

A invertible $\Rightarrow$ $0 \cdot A =$ zero matrix which is not invertible

NOT a subspace

3. One can show that the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbf{R}^3$.

(a) Find the vector $\mathbf{u} \in \mathbf{R}^3$ such that $[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, for the basis $\mathcal{B}$.

$$\vec{u} = (-1)\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 \\ 0+2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) Find the coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ for the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

We need to find x_1, x_2 and x_3 such that

$$x_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \boxed{x_3 = 2}$$

$x_2 + x_3 = 1 \Rightarrow \boxed{x_2 = -1}$

$x_1 + x_2 + x_3 = 1 \Rightarrow \boxed{x_1 = 0}$

$$\Rightarrow [v]_B = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

4. (a) Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 1 & 0 \\ 9 & 0 & 2 \end{bmatrix}$. Calculate the determinant of $(AB)^{-1}$.

$\det A = 3 \cdot 1 \cdot 2 = 6 \qquad \det B = 5 \cdot 1 \cdot 2 = 10$

$\det (AB)^{-1} = \frac{1}{\det(AB)} = \frac{1}{\det A \det B} = \frac{1}{6 \cdot 10} = \frac{1}{60}$

(b) Let A be a 4×5 matrix whose rows are linearly independent. Is it possible that both $[1, 0, 0, 1, 2]^T$ and $[0, 2, 1, 3, 5]^T$ are in the null space of A? Why or why not?

rows Linearly independent $\Rightarrow$ rank $A = 4$

nullity of A + rank A = 5 = # columns

$\Rightarrow \boxed{\text{nullity of } A = 1}$ $\Rightarrow$ all vectors in nullspace multiples of each other

~~If both~~ But $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \\ 5 \end{bmatrix}$ ~~in nullspace $\Rightarrow$ dim nullspace ≥ 2~~ are not multiples of each other

Linearly independent
not multiples of each other

not possible