 Name: ____________________

(6 points) 1. (a) Let $A = \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 3 & 1 & 1 & -1 & h \\ 0 & 1 & -5 & 2 & -5 \end{bmatrix}$. Determine all values of h for which A is *not* full rank.

What is the dimension of the null space of A for those values of h?

$$A \to \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & -5 & 2 & h-9 \\ 0 & 1 & -5 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & 1 & -5 & 2 & h-9 \\ 0 & 0 & 0 & 0 & 4-h \end{bmatrix}$$

If $\boxed{h=4}$ only two pivots, i.e. not full rank

If $h=4$ we have four free parameters

$\Rightarrow \boxed{\text{nullity} = 4}$

(b) Find the solution (in parametric form) of the following system of equations:

$$\begin{aligned} x_1 \quad\quad + 2x_3 &= -1 \\ 3x_1 + x_2 + \; x_3 &= -1 \\ x_2 - 5x_3 &= 2 \end{aligned}$$

Augmented matrix for system

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 3 & 1 & 1 & -1 \\ 0 & 1 & -5 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free parameter

$\Rightarrow \quad x_2 - 5x_3 = 2 \quad \Rightarrow \quad \boxed{x_2 = 2 + 5x_3}$

$x_1 + 2x_3 = -1 \quad \Rightarrow \quad \boxed{x_1 = -1 - 2x_3}$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 - 2x_3 \\ 2 + 5x_3 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \; x_3 \in \mathbb{R}}$$

(6 points) 2. The matrix $A = \begin{bmatrix} 2 & 3 & 1 & 8 & 2 & 3 \\ -1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 1 & -2 & 5 & 3 & 8 \\ 1 & 4 & 3 & 9 & 2 & 1 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find a basis for Col(A).

pivot entries in first, second and fifth column

$\Rightarrow$ basis for Col(A) given by $\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$

(b) Find a basis for Row(A).

First three rows of echelon form (or of A)

$\rightarrow \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

(c) Find a basis for Row$(A)^\perp$.

$Row(A)^\perp = Nul(A)$: ~~with~~

free parameters x_3, x_4, x_6

$x_5 + 2x_6 = 0 \Rightarrow \boxed{x_5 = -2x_6}$

$x_2 + x_3 + 2x_4 - x_6 = 0 \Rightarrow \boxed{x_2 = -x_3 - 2x_4 + x_6}$

$x_1 - x_3 + x_4 + x_6 = 0 \Rightarrow \boxed{x_1 = x_3 - x_4 + x_6}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_3 - x_4 + x_6 \\ -x_3 - 2x_4 + x_6 \\ x_3 \\ x_4 \\ -2x_6 \\ x_6 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{=\vec{b}_1} + x_4 \underbrace{\begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{=\vec{b}_2} + x_6 \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{=\vec{b}_3} \Rightarrow$ ~~basis~~ basis $= \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$

(10 points) 3. In each of the following examples, a vector space V is given, along with a subset $S \subseteq V$. Indicate whether the set S is a subspace of V or not, by filling in the ○ **YES** bubble if it is a subspace, or the ○ **NO** bubble if is not. 2 points will be assigned for each correct response, 1 point for each blank non-response, and 0 points for each incorrect response. No justification is required.

(a) $V = \mathsf{R}^2$, and $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \leq y \right\}$.

○ **YES** ● **NO**

(b) $V = \mathsf{R}^n$, $W \subseteq V$ is a subspace, and $S = W^\perp$.

● **YES** ○ **NO**

(c) $V = \mathsf{R}^3$, and S is the set of vectors in V with length 1: $S = \{\mathsf{v} \in V : \|\mathsf{v}\| = 1\}$.

○ **YES** ● **NO**

$\vec{v} \in S$ but $3\vec{v} \notin S$

(d) $V = M_{4\times 4}$ is the space of 4×4 matrices, and S is the set of *skew-symmetric* matrices: $S = \{A \in M_{4\times 4} : A^\top = -A\}$.

● **YES** ○ **NO**

(e) $V = \mathsf{P}_3$ is the space of polynomials of degree ≤ 3, and $S = \{\mathsf{p} \in \mathsf{P}_3 : \mathsf{p}(1) = 1\}$.

○ **YES** ● **NO**

 Name: ____________________

(8 points) 4. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

(a) Find all the eigenvalues of A. Is A diagonalizable? Explain your answer.

$$\det \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 2 & 2 \\ 1-\lambda & 0 \end{bmatrix}$$

$$= (1-\lambda)\,(1-\lambda)(2-\lambda) - (1-\lambda)\cdot 2$$

$$= (1-\lambda)\left[\lambda^2 - 3\lambda + 2 - 2\right]$$

$$= (1-\lambda)(\lambda^2 - 3\lambda) = \lambda(1-\lambda)(\lambda-3)$$

$\Rightarrow$ eigenvalues $0, 1, 3$

all eigenvalues have multiplicity 1

$\Rightarrow$ A is diagonalizable

(b) Determine the eigenspace of A for the *largest* eigenvalue, 3.

eigenspace = nullspace of $A - 3I = \begin{bmatrix} -2 & 2 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$\rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow -x_2 = 0 \Rightarrow \boxed{x_2 = 0}$

$-x_1 + \underset{\overset{\parallel}{0}}{x_2} + x_3 = 0 \Rightarrow \boxed{x_1 = x_3} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\text{eigenspace} = \left\{ t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ t \in \mathbb{R} \right\}}$ $\rightarrow$ $\boxed{\text{eigenvector } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$

(c) What is the largest eigenvalue of the matrix A^2? And what is its eigenspace?

~~If λ eigenvalue~~ The eigenvalues of A^2 are $\{\lambda^2,\ \lambda$ eigenvalue of $A\}$

$= \{0, 1, 9\}$

eigenspace of eigenvalue 9 (for A^2) same as in b

= eigenspace of eigenvalue 3 for A

$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

 Name: ____________

(7 points) 5. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

(a) Calculate the determinant of A, showing your work. Use this to show that A is invertible.

$\det A = 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 1 - 4 = -3$

$\det A \neq 0 \Rightarrow A$ invertible

(b) Calculate the inverse of A, showing your work.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & -2 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2/3 & -1/3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 \\ 0 & 0 & 1 & 0 & 2/3 & -1/3 \end{array}\right]$$

Result: $\boxed{A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix}}$

(c) Explain why there is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

A is symmetric $\Rightarrow$ orthonormal basis of eigenvectors

$\Uparrow$

Section 7.1 (not required for test)

 Name: ______________

(6 points) 6. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 3 \\ -2 & -1 \end{bmatrix}$.

(a) Use the Gram–Schmidt process to find an orthonormal basis $\{\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_2\}$ for $\mathrm{Col}(A)$.

$$\left\| \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \right\| = \sqrt{1^2+2^2+(-2)^2} = 3 \Rightarrow \boxed{\hat{u}_1 = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \\ -2/3 \end{bmatrix}}$$

$$\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \frac{\begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \frac{9}{9} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\|\vec{u}_2\| = \sqrt{0+1+1+1} = \sqrt{3} \Rightarrow \boxed{\hat{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}}$$

(b) Use the result in part (a) to factorize $A = QR$ where $Q \in M_{4\times 2}$ has orthonormal columns and $R \in M_{2\times 2}$ is upper-triangular.

If $\vec{v}_j$ is the jth column of A, then we have

$$\vec{v}_j = r_{1j}\hat{u}_1 + r_{2j}\hat{u}_2 + \dots + r_{jj}\hat{u}_j$$

hence $\boxed{r_{ij} = \hat{u}_i \cdot \vec{v}_j}$

we calculate $r_{11} = \hat{u}_1 \cdot \vec{v}_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} = 3$

$$r_{12} = \hat{u}_1 \cdot \vec{v}_2 = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} = 3$$

$$r_{22} = \hat{u}_2 \cdot \vec{v}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \sqrt{3}$$

$$\Rightarrow \boxed{A = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/\sqrt{3} \\ 2/3 & 1/\sqrt{3} \\ -2/3 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & \sqrt{3} \end{bmatrix}}$$

(6 points) 7. Suppose A and B are square matrices, and suppose that $AB = BA$. Let v be an eigenvector of A with eigenvalue λ.

(a) Show that $u = Bv$ is also an eigenvector of A, with eigenvalue λ.

$$A\vec{u} = AB\vec{v} = BA\vec{v} = B\lambda\vec{v} = \lambda B\vec{v} = \lambda\vec{u}$$

hence $\vec{u}$ is an eigenvector of A with eigenvalue λ

Remark: One also needs to assume that $\vec{u} \neq 0$, which should have been stated in the problem.

(b) Let v be an eigenvector of A with eigenvalue λ, as above. Suppose that the algebraic multiplicity of the eigenvalue λ is 1. Use (a) to show that v is also an eigenvector of B.

algebraic multiplicity of $\lambda = 1$

$\Rightarrow$ eigenspace of λ is 1-dimensional $\Rightarrow$ spanned by $\vec{v}$

$\Rightarrow$ any eigenvector of A with eigenvalue λ must be a multiple of $\vec{v}$

$\Rightarrow$ $\vec{u} = B\vec{v}$ must be a multiple of $\vec{v}$

i.e. $B\vec{v} = \vec{u} = \alpha\vec{v}$ for some scalar α.

$\Rightarrow$ $\vec{v}$ is eigenvector of B with eigenvalue α