

## MATH 110 MIDTERM 2

**Important: Do not separate these sheets.** Please put your name on each sheet, and your student ID number on the first sheet. Please show *all* your work on the pages provided. These will be uploaded to GradeScope. You may use both sides of each sheet. Use the last sheet if you need more room.

### QUESTION I (20 PTS TOTAL)

Consider the function  $f(x) = \sin^2(\pi x)$  on the interval  $[0, 1]$ .

- (1) (5 pts) Compute the Fourier cosine-series for  $f(x)$  on  $[0, 1]$ .

**Solution:** First we use the basic trig identity  $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ . This gives:

$$f(x) = \frac{1}{2} - \frac{1}{2} \cos(2\pi x) .$$

Since this sum is already of the form of a cosine series we have the answer. One can also compute the coefficients directly using the second identity at the bottom of the page (now on the next page).

**Note:** If you forgot this trig identity, notice that if one differentiates the first trig identity at the bottom of the page with respect to  $\psi$  then one gets  $2 \sin(\theta) \sin(\psi) = \cos(\theta - \psi) - \cos(\theta + \psi)$ . Setting  $\theta = \psi$  and using  $\cos(0) = 1$  gives  $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ .

- (2) (15 pts) Compute the Fourier sin-series for  $f(x)$  on  $[0, 1]$ .

**Solution:** Using the formula for  $f(x)$  above and one integration by parts, followed by the first trig identity at the bottom of the page, we compute:

$$\begin{aligned} a_n &= 2 \int_0^1 \frac{1}{2} (1 - \cos(2\pi x)) \sin(n\pi x) dx = \frac{2}{n} \int_0^1 \sin(2\pi x) \cos(n\pi x) dx , \\ &= \frac{1}{n} \int_0^1 (\sin(\pi(n+2)x) - \sin(\pi(n-2)x)) dx , \\ &= \frac{-1}{n\pi} \left( \frac{1}{n+2} \cos(\pi(n+2)x) \Big|_0^1 - \frac{1}{n-2} \cos(\pi(n-2)x) \Big|_0^1 \right) , \\ &= \frac{-1}{n\pi} \left( \frac{1}{n+2} ((-1)^n - 1) - \frac{1}{n-2} ((-1)^n - 1) \right) = \begin{cases} 0 , & n \text{ even;} \\ \frac{8}{\pi n(4-n^2)} , & n \text{ odd.} \end{cases} \end{aligned}$$

Note that this formula does not make sense for  $n = 2$ , but one can check  $a_2 = 0$  directly at the middle stage of the above calculation. This gives:

$$f(x) = \sum_{n \text{ odd}} \frac{8}{\pi n(4-n^2)} \sin(n\pi x) .$$

**Note:** The above calculation was essentially done in class on October 10 when it was shown that (using the same integration by parts as above):

$$1 - \cos(2x) = \sum_{n \text{ odd}} \frac{16}{\pi n(4-n^2)} \sin(nx) .$$

Hint: For this problem the following identities will be useful,  $2 \sin(\theta) \cos(\psi) = \sin(\theta + \psi) + \sin(\theta - \psi)$  and  $2 \cos(\theta) \cos(\psi) = \cos(\theta + \psi) + \cos(\theta - \psi)$ .

QUESTION II (20 PTS)

A calculation shows that the full Fourier series of  $f(x) = \cosh(x)$  on the interval  $[-1, 1]$  is given by the expression:

$$f(x) = \sinh(1) + 2 \sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + (n\pi)^2} \cos(n\pi x) .$$

Use this information to compute the Fourier sin-series of the function  $f(x) = \sinh(x)$  on the interval  $[0, 1]$ . Please explain carefully any theorems from the text you are using.

**Solution:** Note that the function  $f(x)$  is piecewise differentiable and continuous as a periodic function on  $[-1, 1]$  because  $\cosh(1) = \cosh(-1)$ . Thus, by a theorem in class the Fourier series of  $f'(x) = \sinh(x)$  is given by the term-by-term differentiation of the above formula. That is:

$$f'(x) = \sinh(x) \sim 2 \sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n\pi}{1 + (n\pi)^2} \sin(n\pi x) .$$

Since  $\sinh(x)$  is the odd extension of  $\sinh(x)$  on the interval  $[0, 1]$  the above series is the sin-series for  $\sinh(x)$  on  $[0, 1]$ .

QUESTION III (30 PTS)

Use Questions I and II to solve the following initial value problem for the wave equation on the interval  $[0, 1]$ :

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & u(t, 0) &= u(t, 1) = 0, \\ u(0, x) &= \sin^2(\pi x), & u_t(0, x) &= \sinh(x). \end{aligned}$$

Here  $c > 0$  is a fixed constant.

**Solution:** Recall that if:

$$u(0, x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x), \quad u_t(0, x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x),$$

then the general solution to the above problem is given by:

$$u(t, x) = \sum_{n=1}^{\infty} \left( a_n \cos(cn\pi t) + \frac{b_n}{cn\pi} \sin(cn\pi t) \right) \sin(n\pi x).$$

Plugging the previous two formulas directly into this gives the specific solution:

$$u(t, x) = \sum_{n \text{ odd}} \frac{8}{\pi n(4 - n^2)} \cos(cn\pi t) \sin(n\pi x) + \frac{2}{c} \sinh(1) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{1 + (n\pi)^2} \sin(cn\pi t) \sin(n\pi x).$$

QUESTION IV (30 PTS)

Consider the solution  $u(t, x)$  to the wave equation on  $[0, \pi]$  which is given by the conditions:

$$\begin{aligned} u_{tt} &= u_{xx}, & u(t, 0) &= u(t, \pi) = 0, \\ u(0, x) &= x(\pi - x), & u_t(0, x) &= \sin(x). \end{aligned}$$

For this solution compute the energy at time  $t = 10$  which is given by the formula:

$$E[u]|_{t=10} = \frac{1}{2} \int_0^\pi [u_t^2(10, x) + u_x^2(10, x)] dx.$$

**Solution:** Recall for Dirichlet boundary conditions one has the conservation of energy  $E = \text{const}$ . Thus we may as well compute  $E[u]|_{t=0}$ . This is a direct calculation:

$$E[u]|_{t=0} = \frac{1}{2} \int_0^\pi \sin^2(x) dx + \frac{1}{2} \int_0^\pi (\pi - 2x)^2 dx = \frac{1}{4} \left( x - \frac{1}{2} \sin(2x) \right) \Big|_0^\pi + \frac{1}{12} (2x - \pi)^3 \Big|_0^\pi = \frac{\pi}{4} + \frac{\pi^3}{6}.$$

Note that for the first term we have again used the formula  $\sin^2(x) = \frac{1 - \cos(2x)}{2}$ .

**Note:** The conservation of energy was discussed at length in class, as well as the practice and hand in problems in Section 4.4. You do not need to derive it here.

Use this page to show your work. Please label any additional work here by the corresponding problem number.