Some Solutions for Practice Final 2

Observe : Solutions here may sometimes be a little sketchier than what is expected in the exam. Make sure you justify all your steps.

1. (a) Calculate the coefficients using the formula (for $L = 1$)

$$A_n = 2\int_0^1 x \cos n\pi x \; dx$$

via integration by parts.

(b) Using the fact that $x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x$, we obtain

$$\int_0^1 x^2 dx \;=\; \int_0^1 (\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x)(\frac{A_0}{2} + \sum_{m=1}^{\infty} A_n \cos m\pi x)\; dx. \qquad (*)$$

By orthogonality relations we have

$$\int_0^1 \cos n\pi x \;\; \cos m\pi x \; dx \;=\; \begin{cases} 0 & \text{if } n \neq m, \\ \frac{1}{2} & \text{if } n = m. \end{cases}$$

Hence the right hand side of $(*)$ is equal to

$$\frac{A_0^2}{4} \;+\; \frac{1}{2}\sum_{n=1}^{\infty} A_n^2.$$

Now observe that the left hand side of $(*)$ is equal to $\frac{1}{3}$, and that $A_0 = 1$. Hence we obtain

$$\frac{A_0^2}{4} \;+\; \sum_{n=1}^{\infty} A_n^2 \;=\; 2(\frac{A_0^2}{4} \;+\; \frac{1}{2}\sum_{n=1}^{\infty} A_n^2) - \frac{A_0^2}{4} \;=$$

$$=\; \frac{2}{3} - \frac{1}{2} \;=\; \frac{1}{6}.$$

2. (a) We first calculate separable solutions. It follows from the boundary conditions (fill in the details yourself!) that they are of the form $\sin nx \; e^{-n^2 t}$. It follows from the initial value conditions that the solution is given by

$$w(x,t) \;=\; \sin(2x)e^{-4t} \;+\; \sin(3x)e^{-9t}.$$

(b) The solution is $v(x) = x/\pi$ (check that all conditions are satisfied!)

(c) Let $\tilde{u} = u - v$, with v as in (b). Then we have

$$\tilde{u}(0,t) \;=\; 0, \quad \tilde{u}(\pi,t) \;=\; 0, \quad t > 0,$$

$$\tilde{u}(x,0) \;=\; \sin(2x) + \sin(3x).$$

Moreover, $\tilde{u}$ also satisfies the heat equation. Hence $\tilde{u} = w$, with w as in (a). As $u = \tilde{u} + v$, we obtain the solution

$$u(x,t) \;=\; \sin(2x)e^{-4t} \;+\; \sin(3x)e^{-9t} \;+ x/\pi.$$

3. (a) Solution: $f(r) = J_0(\sqrt{\lambda_n} r)$, where $\lambda_n = \frac{z_{0n}^2}{a^2}$ and z_{0n} is the n-th root of the derivative $J_0'(z)$. *Ask if you do not know how to justify this.*

   (b) Using the orthogonality of the eigenfunctions $v_n(r) = J_0(\sqrt{\lambda_n} r)$ with respect to the inner product

$$(f,g) \;=\; \int_0^a f(r)g(r)rdr,$$

   we obtain

$$A_n = \frac{(r^2, v_n)}{(v_n, v_n)} \;=\; \frac{\int_0^a r^2 J_0(\sqrt{\lambda_n} r) r dr}{\int_0^a J_0^2(\sqrt{\lambda_n} r) r dr}.$$

4. (a) We use the same orthogonality arguments as in Problem 1. Hence we obtain as result

$$(\phi,\psi) \;=\; 2(-1)\frac{1}{2} + (-4)6\frac{1}{2} + 3(0)\frac{1}{2} + (-10)(-2)\frac{1}{2} + 5(3)\frac{1}{2} + (0)(-1)\frac{1}{2} \;=\; \frac{9}{2}.$$

   (b) There was a misprint in the problem: It should be ...r dr instead of dr at the end of the integral. The value of the integral is equal to 0 as the functions are eigenfunctions of $-\Delta = -\nabla^2$ for different eigenvalues, by the theorem mentioned.

5. Let us write $u(x,y) = X(x)Y(y)$. Separating variables, we have

$$X''(x) = -\lambda X, \quad Y''(y) = \lambda Y,$$

   with boundary conditions $X'(0) = 0 = X'(\pi)$. These boundary conditions suggested the choice of the sign for λ It follows (fill in the details!) that

$$X(x) = \cos(nx), \quad \text{with } \lambda = n^2.$$

   For $\lambda = n^2$ the solution for Y is given by

$$Y(y) \;=\; A_n \cosh(ny) \;+\; B_n \sinh(ny).$$

   It follows from the remaining boundary condiitions (justify!)

$$u(x,y) \;=\; \frac{3}{\sinh 2\pi}\cos(2x)\sinh(2y) \;+\; \frac{2}{\sinh 5\pi}\cos(5x)\sinh(5y).$$