1. (a) Let \( R = \mathbb{Z}_4 \), and consider \( 2 \in R \). Then \( 2 \neq 0 \) in \( R \), but \( 2^2 = 4 = 0 \) in \( R \). Hence 2 is a zero divisor in \( R \).

(b) Consider \( f(x) = g(x) = 2x + 1 \in \mathbb{Z}_4[x] \). Then \( \text{deg} f(x) + \text{deg} g(x) = 1 + 1 = 2 \), but \( f(x)g(x) = (2x + 1)^2 = 4x^2 + 4x + 1 = 1 \) in \( \mathbb{Z}_4[x] \), hence \( 0 = \text{deg} f(x)g(x) \leq \text{deg} f(x) + \text{deg} g(x) = 2 \).

(c) Consider the ring of integers \( \mathbb{Z} \). This is clearly an integral domain, but not a field: for example \( 2 \in \mathbb{Z} \) is not invertible in \( \mathbb{Z} \).

(d) Any field \( F \) is an integral domain: by definition \( F \) is a commutative ring with identity. Moreover, let \( a, b \in F \) with \( ab = 0 \). Suppose by contradiction that both \( a \) and \( b \) are not zero. Since \( F \) is a field and \( a \neq 0 \), there exists a multiplicative inverse of \( a \), call it \( a^{-1} \in F \). Multiplying both sides of the equation \( ab = 0 \) by \( a^{-1} \) gives \( b = a^{-1}ab = a^{-1}0 = 0 \), a contradiction. Hence \( F \) is an integral domain.

2. Let \( R \) be an integral domain, and let \( a \in R \) be a nilpotent element. Hence there exists a minimal \( n \in \mathbb{N} \), \( n \geq 1 \) such that \( a^n = 0 \). Assume by contradiction that \( n \geq 2 \). Therefore we have \( a \cdot a^{n-1} = a^n = 0 \), where \( a = a^1 \neq 0 \) and \( a^{n-1} \neq 0 \) because of the minimality of \( n \). But this contradicts our assumption that \( R \) is an integral domain. Hence \( n = 1 \), i.e. \( a = a^1 = 0 \).

3. (a) We have \( f(x)(x - 1) = x^4 - x^3 + x^2 - x + 1 = x^4 - 1 \) and \( (x^2 + 1)(x + 1) = x^3 + x^2 + x + 1 = f(x) \).

(b) \( \langle f(x) \rangle \) is not a prime ideal: \( \langle x^2 + 1 \rangle \) and \( \langle x + 1 \rangle \) are both proper ideals. Moreover, the degrees of \( x^2 + 1 \) and of \( x + 1 \) are less than the degree of \( f(x) \). Hence they can not be multiples of \( f(x) \), i.e. they are not elements of the ideal \( \langle f(x) \rangle \). But by part (a) of this problem \( \langle x^2 + 1 \rangle \langle x + 1 \rangle = f(x) \in \langle f(x) \rangle \).

(c) Setting \( q(x) = x - 1 \), by part (a) of this problem \( q(x)(x^3 + x^2 + x + 1) = (x - 1)f(x) = x^4 - 1 \), i.e. \( x^4 = q(x)(x^3 + x^2 + x + 1) + 1 \), therefore \( r(x) = 1 \).

(d) We perform our calculation in the ring \( R = \mathbb{Z}[x]/\langle x^3 + x^2 + x + 1 \rangle \), denoting by \( g(x) \) the coset of the polynomial \( g(x) \) in \( R \). By part (c) of this problem, \( x^4 = x^4 = (x - 1)(x^3 + x^2 + x + 1) + 1 = \overline{1} \), therefore \( x^{25} + \overline{1} = \overline{x^{25}} + \overline{1} = \overline{x^{24}x + 1} = (\overline{x})^{25} + \overline{1} = \overline{x^{25}} + \overline{1} = \overline{\overline{x} + 1} = \overline{x + 1} \). Therefore the remainder of the division of \( x^{25} + 1 \) by \( x^3 + x^2 + x + 1 \) is \( x + 1 \).

4. (a) We denote by \( \overline{a + bi} \) the coset of the element \( a + bi \in \mathbb{Z}[i] \). Observe that \( (2 + i)(2 - i) = 4 + 1 = 5 \), and \( \overline{7} = \overline{-2} \). Given an element \( a + bi \in \mathbb{Z}[i] \), there exists \( q \in \mathbb{Z} \) and \( r \in \{0, 1, 2, 3, 4\} \) such that \( a - 2b = 5q + r \). Therefore \( a + bi = a - 2b + \overline{r} \). This shows that \( \mathbb{Z}[i]/\langle 2 + i \rangle = \{0, 1, 2, 3, 4\} \).

We want to show that these cosets are distinct. So let \( r, q \) be in \( \mathbb{Z} \) with \( 0 \leq r, q < 5 \) such that \( \overline{r} = \overline{q} \). This means \( r - q \) is in \( \langle 2 + i \rangle \). Hence \( r - q \) is a multiple of \( 2 + i \), which implies that \( 5 = N(2 + i) \) divides \( N(r - q) = (r - q)^2 \); here, as usual, \( N(a + ib) = a^2 + b^2 \) and we use \( N(xy) = N(x)N(y) \) for any
$x, y \in \mathbb{Z}[i]$. But as $|r-q| < 5$, this is possible only if $r = q$. Hence $\mathbb{Z}[i]/\langle 2 + i \rangle$ has exactly five distinct cosets.

(b) If $I$ is an ideal with $\langle 2 + i \rangle \subseteq I$, then $I/\langle 2 + i \rangle$ is, in particular, a subgroup of the group $\mathbb{Z[i]}/\langle 2 + i \rangle$ which has five elements. As 5 is a prime number, this group is isomorphic to $\mathbb{Z}_5$, which has no subgroups except the trivial subgroup and the group itself. Hence $I = \langle 2 + i \rangle$ or $I = \mathbb{Z}[i]$, which shows the claim.

(c) We showed in part (b) that $\langle 2 + i \rangle$ is a maximal ideal. Hence $\mathbb{Z}[i]/\langle 2 + i \rangle$ is a field.