1. (a) Notice that \( p_1(1) = 1 + 4 + 2 = 0 \) in \( \mathbb{Z}_7 \) and \( p_2(2) = 4 + 3 = 0 \) in \( \mathbb{Z}_7 \), hence both polynomials are reducible over \( \mathbb{Z}_7 \). Therefore both \( \mathbb{Z}_7[x]/(p_1(x)) \) and \( \mathbb{Z}_7[x]/(p_2(x)) \) are not fields.

(b) Consider the polynomial \( p(x) = x^2 + 1 \). Observe that \( p(0) = 1 \), \( p(1) = 2 \), \( p(2) = 5 \), \( p(3) = 3 \), \( p(4) = 3 \), \( p(5) = 5 \) and \( p(6) = 2 \) in \( \mathbb{Z}_7 \), hence \( p(x) \) is an irreducible polynomial over \( \mathbb{Z}_7 \) of degree 2. Therefore \( \mathbb{Z}_7[x]/(p(x)) \) is a field of order \( 7^2 = 49 \).

2. (a) Reducing the polynomial \( p(x) = x^3 + 2x + 4 \) modulo 3 gives \( \overline{p}(x) = x^3 - x + 1 \) in \( \mathbb{Z}_3[x] \). Observe that \( \overline{p}(0) = 1 \), \( \overline{p}(1) = 1 \) and \( \overline{p}(2) = 1 \) in \( \mathbb{Z}_3 \), hence \( \overline{p}(x) \) is irreducible over \( \mathbb{Z}_3 \). But this implies that \( p(x) \) is irreducible over \( \mathbb{Q} \).

(b) We can apply Eisenstein criterion with the prime \( p = 2 \) to see that the polynomial \( (x + 1)^3 + 1 = x^4 + 4x^3 + 5x^2 + 4x + 2 \) is irreducible over \( \mathbb{Q} \).

(c) Observe that the polynomial \( p(y) = y^4 + 1 \) is irreducible over \( \mathbb{Q} \) if and only if the polynomial \( p(y + 1) \) is irreducible over \( \mathbb{Q} \). But we have just seen in part (b) of this problem that \( p(y + 1) = (y + 1)^4 + 1 \) is irreducible over \( \mathbb{Q} \) (replace \( y \) by \( x \)), hence \( p(x) \) is irreducible over \( \mathbb{Q} \).

3. Notice that the polynomial \( p(x) = (x-1)^3-2 = x^3-3x^2+3x-3 \) has the property that \( p(a) = 0 \). Applying Eisenstein criterion to \( p(x) \) with the prime \( p = 3 \), we see that \( p(x) \) is irreducible over \( \mathbb{Q} \), hence it must be the minimal polynomial of \( a \) over \( \mathbb{Q} \). Therefore \( \mathbb{Q}(a) \cong \mathbb{Q}[x]/(p(x)) \) is a vector space over \( \mathbb{Q} \) of dimension \( \deg(p(x)) = 3 \). We conclude that \([\mathbb{Q}(a) : \mathbb{Q}] = 3\).

4. (a) We assume that \( y \neq 0 \), hence \( N(y) \neq 0 \). Let \( y = a + b\sqrt{-d} \) with \( a, b \in \mathbb{Z} \). We have \( N(y) = a^2 + b^2d \). If \( b \neq 0 \), then
\[
N(y) = a^2 + b^2d \geq b^2d \geq d \geq 4.
\]
If \( b = 0 \), then \( N(y) = a^2 \). But the only nonzero integer less than 4 which is a square is 1, hence \( N(y) = 1 \).

(b) Suppose by contradiction that 2 is reducible in \( \mathbb{Z}[\sqrt{-d}] \). In this case \( 2 = xy \), where both \( x \) and \( y \) are elements of \( \mathbb{Z}[\sqrt{-d}] \) which are not units. In particular \( N(x) \geq 2 \) and \( N(y) \geq 2 \). But we have just seen in part (a) of this problem that in this case \( N(x) \geq 4 \) and \( N(y) \geq 4 \). This contradicts \( 4 = N(2) = N(xy) = N(x)N(y) \). Therefore 2 is irreducible.

(c) Solution 1: Observe that \( 6 = 2 \cdot 3 = (1 - \sqrt{-5})(1 + \sqrt{-5}) \) in \( \mathbb{Z}[\sqrt{-5}] \). We have already seen in part (b) of this problem that 2 is irreducible. We claim that also 3, \( 1 - \sqrt{-5} \) and \( 1 + \sqrt{-5} \) are irreducible.

Suppose the contrary, say 3 is reducible. Hence we have \( 3 = xy \) where both \( x \) and \( y \) are elements of \( \mathbb{Z}[\sqrt{-d}] \) which are not units. In particular \( N(x) \geq 2 \) and \( N(y) \geq 2 \). Since \( 9 = N(3) = N(xy) = N(x)N(y) \), we should have \( N(x) = N(y) = 3 \). But this contradicts part (a) of this problem. Hence 3 is irreducible in \( \mathbb{Z}[\sqrt{-5}] \).
Similarly, if $1 + \sqrt{-5}$ is reducible, then $1 + \sqrt{-5} = xy$ where both $x$ and $y$ are elements of $\mathbb{Z}[\sqrt{-5}]$ which are not units. In particular $N(x) \geq 2$ and $N(y) \geq 2$. Since $6 = N(1 + \sqrt{-5}) = N(xy) = N(x)N(y)$, we must have $N(x) = 2$ and $N(y) = 3$, or vice versa $N(x) = 3$ and $N(y) = 2$. In both cases this contradict part (a) of this problem. Hence $1 - \sqrt{-5}$ and $1 + \sqrt{-5}$ are irreducible in $\mathbb{Z}[\sqrt{-5}]$.

To show that the unique factorization property does not hold for $6 \in \mathbb{Z}[\sqrt{-5}]$, it remains to show that $2$ is not an associate of $1 - \sqrt{-5}$ or $1 + \sqrt{-5}$. But this follows immediately from the fact that $4 = N(2) \neq N(1 - \sqrt{-5}) = N(1 + \sqrt{-5}) = 6$.

**Solution 2:** Here is a slightly trickier solution which avoids having to prove irreducibility also for $3$ and $1 \pm \sqrt{-5}$:

We have $2 \cdot 3 = 6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$. We already know that $2$ is irreducible. If $1 \pm \sqrt{-5}$ were not irreducible, we could write each of them as a product of irreducible elements, say $1 + \sqrt{-5} = a_1 \ldots a_r$ and $1 - \sqrt{-5} = b_1 \ldots b_s$.

If the factorization property held in $\mathbb{Z}[\sqrt{-5}]$, $2$ would have to be associate to one of these factors, say $a_1$ for simplicity. Hence $2 = a_1u$ for some unit, and $4 = N(2) = N(a_1)N(u) = N(a_1)$. But as $a_1$ divides $1 + \sqrt{-5}$, this would imply $4 = N(a_1)$ divides $N(1 + \sqrt{-5}) = 6$, a contradiction.

(d) We know that $2$ divides $(1 - \sqrt{-5})(1 + \sqrt{-5}) = 6 = 2 \cdot 3$, but $2$ cannot divide $1 - \sqrt{-5}$ or $1 + \sqrt{-5}$ since $N(2) = 4$ does not divide $6 = N(1 - \sqrt{-5}) = N(1 + \sqrt{-5})$. Hence $2$ is not prime in $\mathbb{Z}[\sqrt{-5}]$. 