ON TENSOR CATEGORIES OF LIE TYPE $E_N$, $N \neq 9$

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Abstract. Let $V$ be the minuscule representation of the Lie algebra $\mathfrak{g}(E_N)$, $N = 6, 7$. We show that the centralizer of the action of the quantum group $U_q\mathfrak{g}(E_N)$ on $V^\otimes n$ is generated by the $R$-matrices and one additional element, appearing in the $(N-1)^{st}$ tensor power; a similar description can be given for the classical limit. An analogous statement is true for a certain direct summand of $V^\otimes n$ for $U_q\mathfrak{g}(E_N)$ with $N > 5, N \neq 9$; here $V$ is the module whose highest weight is the fundamental weight labeled by the vertex furthest from the triple point of the graph $E_N$. Moreover, we obtain a 2-parameter family of braid representations generalizing a $q$-version of the Brauer centralizer algebras.

A conceptual description of the centralizer algebras $\text{End}_G(V^\otimes n)$ has been known for the vector representations of the general linear groups (Schur) and of orthogonal and symplectic groups (Brauer) for some time. This played a crucial role in determining all $G$-invariant polynomial maps on $V$. This problem has also been solved for the smallest representation of $\mathfrak{g}(G_2)$ and a few more representations (see [25]) of classical Lie groups. Not that much more seems to be known for other representations and other Lie types.

In this paper, we determine a minimal set of generators for $\text{End}_G(V^\otimes n)$ for the minuscule representations $V$ for Lie types $E_6$ and $E_7$. This will be done in the setting of the Drinfeld-Jimbo quantum groups $U = U_q\mathfrak{g}(E_N)$, and also includes the solution for the classical case for $q = 1$. The quantum case turns out to be more manageable due to the representations of braid groups, via $R$-matrices, in the centralizer of the action of $U$ on $V^\otimes n$. We actually consider the corresponding braid action on a path space. For type $A$ this would be the $q$-analog of Young’s orthogonal representation of the symmetric group for Hecke algebras (see [9], [31]); similar studies also appeared in connection with the star triangle equation in connection with statistical mechanics (see e.g. [11]). These representations are very suitable for determining whether the image of the braid group generates the whole centralizer; whenever this is not the case, we get a new generator for $\text{End}_U(V^\otimes n)$. This program can be fully carried out for a certain summand $V^\otimes n_{new}$ of $V^\otimes n$; here $V$ is the fundamental representation of the Kac-Moody algebra $\mathfrak{g}(E_N)$, $N \neq 9$ (or its quantum version) labeled by the vertex furthest from the triple point of the graph $E_N$. The terminology comes from the fact that for finite $E_N$ (including $N = 8$) it contains all the simple summands of $V^\otimes n$ which have not already appeared in smaller tensor powers; for $N > 9$ the term is a bit of a misnomer as now this summand contains all simple representations which will not appear in any higher tensor powers of $V$. At least for these special summands, one observes a similar uniform decomposition behaviour.

*Supported in part by NSF grants.
for our $E$-series as it is known for the classical type $A$ and type $BCD$ series. While this is different from the proposed exceptional series by Vogel and Deligne, it may be helpful for their studies.

Let us now give a more detailed description of the contents of this paper. Roughly speaking, it contains a combinatorial part and a more algebraic part. As already mentioned, we study the ‘new’ part of $V^\otimes n$. This can be considered a version of Weyl’s approach of restricting the centralizer algebra of orthogonal groups to traceless tensors. One of the obvious problems is to determine, for a given dominant weight $\lambda$, the smallest number $n = n(\lambda)$ for which the simple highest weight module $V_\lambda$ appears in $V^\otimes n$. This problem is solved for $N < 9$ in Section 2; the same formula also gives the largest number $n$ for which $V_\lambda$ appears in $V^\otimes n$ in the Kac-Moody case $N > 9$. Moreover, this leads to a uniform labeling of the dominant weights for all Kac-Moody types $E_N$, $N \neq 9$, which is also done in Section 2. This gives a combinatorial definition of a summand $V^\otimes n_{\text{new}}$ also in the Kac-Moody case. We also observe generic patterns in the tensor product rules for these summands. This can be checked comparatively easily using Littelmann’s path formalism.

A second factor in the definition of the generic labeling for dominant weights was the gradation of a $\mathfrak{g}(E_N)$ module in terms of $\mathfrak{g}(D_{N-1})$ submodules. One observes easily that $\text{End}_{\mathcal{U}(V^\otimes n)}$ contains as a quotient algebra $\text{End}_{\mathcal{U}(\mathfrak{g}(D_{N-1}))}(\overline{V}^\otimes n)$, where $\overline{V}$ is the vector representation of $\mathcal{U}(\mathfrak{g}(D_{N-1}))$. This shows that besides the $R$-matrices we need at least one more generator, an analogue of the Pfaffian in the $(N - 1)^{st}$ tensor power. Our main result shows that for $N = 6, 7$ we need no more generators.

To prove this we study the representations of the braid groups on the Littelmann paths. Here the matrices have a natural block structure coming from the path formalism. To determine these blocks one uses Drinfeld’s quantum Casimir in the form of $q$-Jucys-Murphy elements (see also e.g. [17], [24], [18]). It turns out that in each of these blocks at least one of the eigenidempotents has rank 1, and that the action of the braid generator is determined by the action of this idempotent (this was already observed by Leduc and Ram [18] in the context of the $q$-Brauer algebra). The idempotent itself can now, basically, be determined by additional equations coming from the eigenidempotent property. In particular, the question whether the braid group generates the whole centralizer depends on whether the diagonal entries of these idempotents are zero or not; these entries are special cases of $q$-$6j$-symbols. The basic equations and criteria for nonzero diagonal entries are derived in Section 4. It also contains the proof of the main theorem about $E_6$ and $E_7$ in the quantum case. Explicit formulas for the diagonal entries are proved in Section 5. The main theorem for the classical case can then be easily derived from the quantum case. Moreover, one also obtains a 2-parameter family of braid representations, defined over the field $\mathbb{Q}(r, q)$ of rational functions in 2 variables which generalizes the $q$-Brauer algebra found in [1] and [21]. These representations should be useful for further studies of tensor categories of exceptional Lie type. Some of the possible applications are discussed at the end of this paper and in [34], where several of the results of this paper were announced.
1. Preliminaries

Let \( \mathfrak{g} \) be a symmetrizable Kac-Moody algebra given by a Coxeter graph \( X \) with \( N \) vertices, with generators \( e_i \) and \( f_i, 0 \leq i \leq N - 1 \). We denote the simple roots by \( \alpha_i, i = 0, \ldots, N - 1 \). Fix an invariant bilinear form \( \langle , \rangle \) on \( \mathfrak{h}^* \), and define \( \tilde{\alpha}_i = \frac{2}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \). If all the roots have the same length, we assume \( \langle , \rangle \) to be normalized such that \( \tilde{\alpha}_i = \alpha \). If \( \langle , \rangle \) is nondegenerate, we define the fundamental weights \( \Lambda_j \) by \( \langle \tilde{\alpha}_i, \Lambda_j \rangle = \delta_{ij} \). If \( \alpha \in \mathfrak{h}^* \), the reflection \( s_\alpha \) on \( \mathfrak{h}^* \) is defined by \( s_\alpha(\lambda) = \lambda - \langle \lambda, \tilde{\alpha}_i \rangle \alpha \).

1.1. Gradation via Lie subalgebra. Let \( \mathfrak{g}_0 \) be a Lie subalgebra of \( \mathfrak{g} \) corresponding to the graph obtained from \( X \) by removing the vertex labeled by \( 0 \). Let \( V \) be an irreducible highest weight \( \mathfrak{g} \) module with highest weight \( \Lambda \). We denote by \( V[\mu] \) its weight space corresponding to the weight \( \mu \). Moreover, if \( \mu \) is a weight of \( \mathfrak{g}_0 \), we define \( \tilde{\mu} \) to be the weight of \( \mathfrak{g}_0 \) obtained by restricting \( \mu \) to the coroot lattice of \( \mathfrak{g}_0 \) which we can identify with the \( \mathbb{Z} \)-span of \( \tilde{\alpha}_i \), \( 1 \leq i \leq N - 1 \). We define for any \( i = 0, 1, \ldots \) the level \( i \) subspaces for tensor powers of \( V \) by

\[
V^{\otimes n}[i] = \text{span}\{V[\mu], \langle n\Lambda - \mu, \Lambda_0 \rangle = i\}.
\]

Conversely, we say that a weight \( \mu \) has level \( i \) in \( V^{\otimes n} \) if \( V^{\otimes n}[\mu] \subset V^{\otimes n}[i] \); in this case we denote the level of \( \mu \) by \( \text{lev}_n(\mu) \) or just \( \text{lev}(\mu) \) if no confusion arises.

**Remark.** If \( \mathfrak{g}_0 \) is the classical part of an affine Kac-Moody algebra, and \( V \) a highest weight representation, the spaces we call level spaces are usually called energy spaces.

**Lemma 1.1.** (a) Each level space \( V^{\otimes n}[i] \) is a \( \mathfrak{g}_0 \)-module.

(b) \( V^{\otimes n}[0] \cong V^{\otimes n}_\Lambda \) as a \( \mathfrak{g}_0 \)-module. In particular, the space of \( \mathfrak{g} \)-intertwiners \( \text{End}_\mathbb{C}(V^{\otimes n}) \) contains a subalgebra isomorphic to \( \text{End}_{\mathfrak{g}_0}(V^{\otimes n}_\Lambda) \), and \( \text{mult}_{V[\mu]}(V^{\otimes n}) = \text{mult}_{V[\tilde{\mu}]}(V^{\otimes n}_\Lambda) \) for any weight \( \mu \) with \( \text{lev}_n(\mu) = 0 \).

(c) If \( \lambda \) has level \( i \) in \( V^{\otimes n} \) and \( \lambda = \sum_{j=1}^n \omega_j \), with \( \omega_j \) a weight in \( V \), for \( j = 1, 2, \ldots, n \), then \( i \) is equal to the sum of the levels of \( \omega_j \) in \( V \).

**Proof.** If \( \mu \) has level \( i \) in \( V^{\otimes n} \), then so has \( \mu \pm \alpha_j \) for \( j > 0 \). Hence if \( v \) is a weight vector of weight \( \mu \), \( v \) as well as \( e_jv \) and \( f_jv \) are in \( V[\tilde{\tau}] \). This shows (a).

By definition of level spaces, any level 0 vector is killed by \( e_0 \). Hence any \( \mathfrak{g}_0 \) highest weight vector \( v \in V^{\otimes n}[0] \) is also a highest weight vector with respect to \( \mathfrak{g} \). Statement (b) follows from this. Using the notation \( \text{lev}(\omega) = \text{lev}_1(\omega) \) for the level of a weight \( \omega \) of \( V \), we obtain

\[
i = \langle n\Lambda - \mu, \Lambda_0 \rangle = \sum_{j=1}^n \langle \Lambda - \omega_j, \Lambda_0 \rangle = \sum \text{lev}(\omega_j).
\]

1.2. Littelmann paths. To get more precise information about weight multiplicities and tensor product rules we will use Littelmann paths (see [19]); Littelmann calls these paths \( L.S. \) paths in honor of Lakshmibai and Seshadri. There are other methods available; indeed for decomposing tensor products of minuscule and adjoint representations of semisimple Lie algebras (which would suffice for a major part of this paper) methods already known to Brauer would be sufficient (see e.g. [26], Chapter 7 for theorems and history).

Let \( \lambda \) be a dominant integral weight for \( \mathfrak{g} \) and let \( V^\lambda \) be the corresponding irreducible highest weight module. Then Littelmann showed that there exists a labeling set for a basis
for \(V_\lambda\) consisting of paths which can be obtained by applying certain root operators \(f_\alpha\) to the straight line from 0 to \(\lambda\); here \(\alpha\) is a simple root.

Here is an outline of the definition of \(f_\alpha\) (see [19] for the precise definition). If \(\pi : [0, 1] \to \mathfrak{h}^*\) is a piecewise linear path in \(\mathfrak{h}^*\), we define the height function \(h_\alpha(t)\) by \(h_\alpha(t) = \langle \pi(t), \alpha \rangle\). Let \(m_\alpha = \min h_\alpha(t)\). If \(h_\alpha(1) - m_\alpha < 1\), we define \(f_\alpha \pi = 0\). Otherwise, let \(t_0 = \max \{t, h_\alpha(t) = m_\alpha\}\) and let \(t_1 = \min \{t, h_\alpha(t) = m_\alpha + 1\}\). Then \(f_\alpha \pi\) is the path obtained by reflecting \(\pi([t_0, t_1])\) in the hyperplane given by \((x, \alpha) = m_\alpha\) and by defining \(f_\alpha \pi(t) = \pi(t) - \alpha\) for \(t \in [t_1, 1]\). Similarly, one defines root operators \(e_\alpha\), which we will not do here in detail. They have the nice property that \(e_\alpha f_\alpha(\pi) = \pi\) for all paths \(\pi\) for which \(f_\alpha \pi \neq 0\), and that the algebra generated by root operators acts irreducibly on the span of weight paths if and only if the corresponding Lie algebra action is irreducible. Certain additional subtleties in the definitions in [19] will not be relevant here. The reader who wants to get a feel for path bases is invited to prove statements (a) and (b) below. We define for a weight \(\omega\) of \(V_\lambda\) the path \(\pi_\omega\) to be the straight line from 0 to \(\omega\).

(a) Let \(V\) be a minuscule module, i.e. all the weights of \(V\) are conjugate to the highest weight. This implies that \((\omega, \alpha) \in \{\pm 1, 0\}\) for all simple roots \(\alpha\). Then the path basis for \(V\) is given by \(\{\pi_\omega\}\) with \(\omega\) running through the weights of \(V\).

(b) If \(V\) is the adjoint representation of a simply laced Lie algebra, then the basis is labeled by \(\{\pi_\alpha\} \cup \{\pi_i\}\); here \(\alpha\) runs through the roots of \(\mathfrak{g}\), and \(\pi_i\) is the piecewise linear path going from 0 to \(-1/2\alpha_i\) and back to 0, where \(\alpha_i\) is a simple root.

**Lemma 1.2.** Assume the notations of Section 1.1. Moreover, let \(\mathfrak{g}_{-1}\) be the Lie subalgebra of \(\mathfrak{g}_0\) whose generators belong to vertices not connected to vertex 0. Then the highest weight vectors in the level space \(V[i]\) (with respect to \(\mathfrak{g}_0\)) are labeled by paths \(\pi\) satisfying

1. \(\pi = f_{a_0} \pi'\), where \(\pi'\) is a highest weight path in \(V[i-1]\) with respect to \(\mathfrak{g}_{-1}\), and
2. \(e_{a_j} \pi = 0\) for all simple roots \(a_j\) of \(\mathfrak{g}_0\) which are not in \(\mathfrak{g}_{-1}\) (i.e. for which the generators \(e_j, f_j\) are not in \(\mathfrak{g}_0\)).

**Proof.** If \(\pi\) labels a highest weight vector with respect to \(\mathfrak{g}_0\) which is not the highest weight vector for \(\mathfrak{g}\), then \(\pi' = e_{a_0} \pi\) can not be the zero path and \(f_{a_0} \pi' = f_{a_0} e_{a_0} \pi = \pi\) (see [19]). Property (ii) follows as \(e_{a_0}\) commutes with the root operators coming from \(\mathfrak{g}_{-1}\). The other implication is shown similarly.

**Remark 1.3.** In practice, it suffices to consider only those highest weights \(\nu\) in \(V[i-1]\) (with respect to \(\mathfrak{g}_{-1}\)) for which \(\nu - a_0\) is a dominant weight for \(\mathfrak{g}_0\). The latter can often be quite easily checked using the fact that \(e_{a_0} \pi = 0\) if \(m_\alpha > -1\).

For 2 given piecewise linear paths \(\pi_1 : [0, n] \to \mathfrak{h}^*\) and \(\pi_2 : [0, m] \to \mathfrak{h}^*\) we define the concatenation \(\pi_1 * \pi_2 : [0, n + m] \to \mathfrak{h}^*\) by

\[
(1.1) \quad \pi_1 * \pi_2(t) = \begin{cases} 
\pi_1(t) & \text{if } 0 \leq t \leq n, \\
\pi_1(n) + \pi_2(t-n) & \text{if } n \leq t \leq n + m.
\end{cases}
\]

Paths \(\pi\) representing basis vectors of \(V\) will always be parametrized via the interval \([0, 1]\). For a given module \(V\), we define a path of length \(n\) to be any piecewise linear path which can be
obtained by concatenating \( n \) basis paths. It is parametrized by the interval \([0, n]\). We refer to subsets \( \pi([i, i + 1]) \), \( i = 0, 1, \ldots \) of the image of a path \( \pi \) as \textit{segments} of \( \pi \). Then we have

**Theorem 1.4.** ([19]) The multiplicity of a highest weight module \( V_{\lambda} \) in \( V^\otimes n \) is equal to the number of paths of length \( n \) terminating at \( \lambda \) which are entirely in the closure of the dominant Weyl chamber.

The last theorem suggests the following definitions, all with respect to a fixed simple highest weight module \( V \): Let \( \mathcal{P}_n \) denote the set of all paths of length \( n \) within the closure of the dominant Weyl chamber, and let \( \mathcal{P}_n(\mu) \) be the subset of all those paths in \( \mathcal{P}_n \) which end in \( \mu \). It is easy to see that for \( V \) the vector representation of \( GL(N) \), a path of length \( n \) corresponds to a Young tableau of a Young diagram \( \mu \) with \( n \) boxes. For this, and notational reasons, we shall in future denote paths in \( \mathcal{P}_n \) by letters \( t, s, \ldots \).

**Corollary 1.5.** There exists an assignment \( t \in \mathcal{P}_n \mapsto p_t \in C_n = \text{End}_\mathfrak{g}(V^\otimes n) \) such that \( p_t V^\otimes n \) is an irreducible \( \mathfrak{g} \)-module with highest weight \( t(n) \), and such that \( p_t p_s = \delta_{ts} p_t \).

One checks easily that \( z^{(n)}_\mu = \sum_{t \in \mathcal{P}_n(\mu)} p_t \) is a central idempotent in \( C_n = \text{End}_\mathfrak{g}(V^\otimes n) \). One also checks that for given \( s \in \mathcal{P}_{n-1} \) we have

\[
p_s = \sum_{t, t' = s} p_t,
\]

where \( t' = t|_{[0, n-1]} \). Consider the subalgebra \( C_{n-1} \otimes 1 \subset C_n \); if no confusion arises we will usually only denote the latter algebra by \( C_{n-1} \). Let \( W^{[n]}_\mu \) be a simple \( C_n \)-module labeled by the dominant weight \( \mu \). Then we have the following isomorphism of \( C_{n-1} \)-modules:

\[
W^{[n]}_\mu \cong \bigoplus_\lambda m_\lambda W^{(n-1)}_\lambda,
\]

where \( \lambda \) runs through all highest weights in \( V^\otimes (n-1) \), and the multiplicity \( m_\lambda \) of the simple \( C_{n-1} \)-module \( W^{(n-1)}_\lambda \) is equal to the number of paths of length \( 1 \) from \( \lambda \) to \( \mu \).

By definition, we can define a basis \( (v_t)_{t \in \mathcal{P}_n(\mu)} \) for the simple \( C_{n-1} \)-module \( W^{(n)}_\mu \) such that \( v_t \) spans the image of \( p_t \) for each \( t \in \mathcal{P}_n(\mu) \). Let \( \delta, \mu \) be dominant weights for which \( V_\delta \subset V^\otimes n-k \) and \( V_\mu \subset V^\otimes n \), and let \( \mathcal{P}_k(\delta, \mu) \) be the set of all paths of length \( k \) from \( \delta \) to \( \mu \) within the closure of the dominant Weyl chamber. Let \( W(\delta, \mu) \) be the vector space spanned by these paths. Then we obtain a representation of \( C_k = \text{End}_\mathfrak{g}(V^\otimes k) \) on \( W(\delta, \mu) \) by

\[
a \in \text{End}(V^\otimes k) \mapsto (p_t \otimes a) z^{(n)}_\mu;
\]

here we used the obvious bijection between elements \( s \in \mathcal{P}_k(\delta, \mu) \) and paths \( \bar{s} \in \mathcal{P}_n(\mu) \) for which \( \bar{s}|_{[0, n-k]} = t \).

1.3. Quantum groups. Let \( U = U_q \mathfrak{g} \) be the Drinfeld-Jimbo quantum deformation of the universal enveloping algebra of \( \mathfrak{g} \); for the context of this paper, we also assume \( \mathfrak{g} \) to be a simply-laced Kac-Moody algebra with invertible Cartan matrix. In this case, most definitions of \( U_q \mathfrak{g} \) coincide, so we can e.g. assume the version of the quantum group as in Lusztig's book.
[20], with \( q = v \). We assume as ground ring the field \( \mathbb{Q}(q) \) of rational functions in the variable \( q \); most of the results hold in greater generality (e.g., if \( q \) is a complex number not equal to a root of unity or 0). It is well-known that in our setting the category \( \text{Rep}(U) \) of integrable representations of \( U \) is semisimple, and it has the same Grothendieck semiring as the original Lie algebra. Moreover, \( \text{Rep}(U) \) is a braided tensor category. This implies that for \( U \)-modules \( V, W \), there are natural braiding isomorphisms \( R_{VW} : V \otimes W \to W \otimes V \) which satisfy

\[
R_{U \otimes V,W} = (R_{UV} \otimes 1_V)(1_U \otimes R_{VW}),
\]

where \( U, V, W \) are \( U \)-modules. Let \( B_n \) be Artin’s braid groups, given by generators \( \sigma_i, 1 \leq i \leq n-1 \) and relations \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \) as well as \( \sigma_i \sigma_j = \sigma_j \sigma_i \) for \( |i - j| \geq 2 \). We obtain, for any \( U \)-module \( V \), a representation of Artin’s braid group \( B_n \) in \( \text{End}(V^{\otimes n}) \) by the map

\[
\sigma_i \mapsto R_i = 1_{i-1} \otimes R_{VV} \otimes 1_{n-1-i} \in C_n = \text{End}(V^{\otimes n}),
\]

where \( 1_j \) is the identity map on \( V^{\otimes j} \).

As the Grothendieck semiring of \( \text{Rep}(U q \mathfrak{g}) \) coincides with the one of \( \text{Rep}(\mathfrak{g}) \), we can also use the Littelmann paths for the decomposition of tensor products of representations of \( U q \mathfrak{g} \). Hence we can represent \( C_n \) on a vector space whose basis is labeled by Littelmann paths (see Section 1.2). This induces, in particular, another representation of \( B_n \), given by

\[
\sigma_i \mapsto A_i = t \to \sum_s a^{(i)}_{st} s
\]

If \( t \in \mathcal{P}_n(\mu) \), then so are the paths \( s \) in the equation above. Moreover, it follows from Eq. 1.3 that the paths \( s \) differ from \( t \) only in the interval \([i-1, i+1]\). Because of this we shall often only consider spaces \( W_i(\lambda, \nu) \) with a basis consisting of paths of length 2 from \( \lambda = t(i-1) \) to \( \nu = t(i+1) \). Equation 1.5 induces an obvious action of \( A_i \) on \( W_i(\lambda, \nu) \). We will call the corresponding matrix block \( A_i(\lambda, \nu) \). If there is no danger of confusion, we will often suppress the index \( i \) in \( W_i(\lambda, \nu) \) and \( A_i(\lambda, \nu) \). We shall also need the following theorem, due to Drinfeld [8].

**Proposition 1.6.** Let \( V_\lambda, V_\nu, V_\Lambda = V \) be simple \( U \)-modules with highest weights \( \lambda, \mu, \Lambda \) respectively, and such that \( V_\mu \) is a submodule of \( V_\lambda \otimes V_\Lambda \). Then

\[
(R_{V_\lambda, V_\Lambda} R_{V_\Lambda, V_\lambda})|_{V_\mu} = q^{\epsilon_\mu - \epsilon_\lambda - \epsilon_\Lambda} 1_{V_\mu},
\]

where for any weight \( \gamma \) the quantity \( \epsilon_\gamma \) is given by \( \langle \gamma + 2\rho, \gamma \rangle \).

Observe that the braiding axiom also implies that the braiding matrices \( R_{V^{\otimes n-1}, V} \) and \( R_{V^{\otimes n-1}, V} \) are equal to \( R_1 R_2 ... R_{n-1} \) and \( R_{n-1} ... R_2 R_1 \) respectively. Using the notation introduced before the last proposition, we obtain

**Corollary 1.7.** The element \( M_n = A_n A_{n-1} ... A_1^2 ... A_1 A_n \) acts on the path \( t \) via the scalar \( q^{\epsilon(\lambda) - \epsilon(\Lambda) - \epsilon(\Lambda)} \), where \( \Lambda \) is the highest weight of \( V \).
1.4. Involutions. We fix a simple highest weight module \( V = V_\lambda \) of the quantum group \( U = U_q \mathfrak{g} \). There exists an algebra homomorphism \( \xi /1/ \mathbb{Q}(q) \) uniquely determined by \( \xi b = b \) for \( b \in \mathbb{Q} \) and by \( \xi q = q^{-1} \). Let \( \mathcal{C}_n = \text{End}_U(V^\otimes n) \). It has been shown by Kirillov jr [16] (see also [33]) that there exists an involutive functorial antiautomorphism \( \ast \) on \( \mathcal{C}_n \); this means \( \ast \) has the properties

- \((a^*)^* = a \) for all \( a \in \mathcal{C}_n \),
- \((a \circ b)^* = a^* \circ b^* \), for \( a \in \mathcal{C}_n \), \( b \in \mathcal{C}_n \) with \( n_1 + n_2 = n \),
- \((f b a)^* = \bar{f} b^* a^* \), for \( a, b \in \mathcal{C}_n \) and \( f \in \mathbb{Q}(q) \).

One can also show that \( z^*_\lambda = z_\lambda \) for any central idempotent \( z_\lambda \in \mathcal{C}_n \). In particular, if the path idempotent \( p_i \) can be obtained as \( p_i = \prod_{i=1}^n z_{(i)} \), we also have \( p_i^* = p_i \).

Similarly, we can define a linear antiautomorphism \( T \) on \( \mathcal{C}_n \) using Kashiwara's inner product. In the following we use the notations for quantum groups as in [20], Section 3, with some minor modifications: We denote the generators of \( U \) by \( e_i, f_i, k_i^{\pm 1} \). As we are in the simply laced case, we can set \( q = v \) and \( k_i = k_i \) for all indices \( i \). Then one checks (see [20], 19.1.1) that the maps

\[
e_i \mapsto q f_i k_i, \quad f_i \mapsto q e_i k_i^{-1}, \quad k_i \mapsto k_i,
\]

extend to a unique antiautomorphism \( \rho \) from \( U \) into itself. If \( \Delta \) is the coproduct of \( U \), defined by

\[
\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i, \quad \Delta(k_i) = k_i \otimes k_i,
\]

it is straightforward to check for the generators (and hence for the whole algebra \( U \)) that

\[
\Delta \circ \rho = (\rho \otimes \rho) \circ \Delta \tag{*}
\]

It is shown (see e.g., [20], 19.1.2) that there exists an up to scalar multiples unique symmetric bilinear form \( \langle \cdot , \cdot \rangle \) on the highest weight \( U \)-module \( V \) such that \( \langle u v , w \rangle = \langle v , u \rangle w \) for all \( u, v, w \in V \) and \( u \in U \). Extending the bilinear form \( \langle \cdot , \cdot \rangle \) to \( V^\otimes n \) in the obvious way, the same equation holds as well for \( u, v, w \in V^\otimes n \), by (\ast\ast)\,. If \( a^T \) denotes the adjoint of \( a \in \text{End}(V^\otimes n) \) with respect to \( \langle \cdot , \cdot \rangle \), we therefore obtain \( a^T = \rho(u) \in U \) for \( u \in U \); here we identify \( u \) with the linear operator by which it acts on \( V^\otimes n \). Hence, if \( c \in \mathcal{C}_n \), it also follows that \( c^T \in \mathcal{C}_n \).

Moreover, it follows from the definition of \( \langle \cdot , \cdot \rangle \) on \( V^\otimes n \) that the operation \( T \) is functorial, i.e., \((c_1 \circ c_2)^T = c_1^T \otimes c_2^T \) for \( c_1 \in \mathcal{C}_{n_1} \), \( c_2 \in \mathcal{C}_{n_2} \), \( n_1 + n_2 = n \), and it obviously is an antiautomorphism. As \( \langle \cdot , \cdot \rangle \) is symmetric, one also deduces easily that \( T \) is involutive, i.e., \((a^T)^T = a \) for all \( a \in \text{End}(V^\otimes n) \). We can now deduce the following Theorem, of which at least part (a) has already been known for the Hermitian case (see [16]).

Theorem 1.8. (a) There exist involutive functorial antiautomorphisms \( * \) and \( T \) on \( \mathcal{C}_n \), with \( * \) antilinear and \( T \) linear.

(b) Both \( * \) and \( T \) act trivially on central idempotents of \( \mathcal{C}_n \), and \( R_i^2 = R_i \) and \( R_i^* = R_i^{-1} \) for \( 1 \leq i \leq n-1 \).
Lemma 1.9. Consider the representation of $C_n$ on the path space. Moreover, assume that $V \otimes^2$ decomposes as a direct sum of mutually nonisomorphic modules. Then

(a) The basis vectors can be normalized over an algebraic extension of $\mathbb{Q}(q)$ in such a way that any idempotent $p_i$ of $A_i$, as well as $A_i$ itself, act as a symmetric matrix.

(b) For any idempotent $P_i$ of $A_i$, and any path idempotent $p_i$ we have $p_i P_i p_i = \alpha_i p_i$, with $\alpha \in \mathbb{Q}(q)$ such that $\bar{\alpha} = \alpha$.

Proof. Part (b) follows from the fact that $p_i P_i p_i$ and $p_i$ are invariant under $\ast$. For part (a), observe that the space $W_\lambda$ spanned by highest weight vectors of $V \otimes^2$ with weight $\lambda$ is a simple $C_n$-module $W_\lambda$ (assuming it is nonzero). Hence we can define an orthogonal basis $(v_i)_{i \in \mathcal{P}_n(\lambda)}$ with respect to the inner product on $V \otimes^2$, where $v_i \in p_i W_\lambda$. After adjoining suitable square roots to $\mathbb{Q}(q)$, we can make this into an orthonormal basis. By construction, the operation $T$ coincides with the transpose of the matrices acting on our vector space. By our assumptions, each spectral idempotent of $A$ is a central idempotent in $C_2$; hence it is invariant under $T$, and so is $A$. This shows part (a).

Remark. Part (a) of the last lemma holds for arbitrary $U_q\mathfrak{g}$-modules. This can be derived from the explicit formula of the $R$-matrix. For finite-dimensional $\mathfrak{g}$, one can always find a module $V$ generating the representation ring of $U_q\mathfrak{g}$ satisfying the condition in (a); the general case can be deduced from this using the braiding axioms.

1.5. The example $D_N$. We identify the Cartan algebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ of Lie type $D_N$ with $\mathbb{R}^N$, with the pairing given by the usual inner product $\langle \lambda, \rho \rangle$ which makes the standard basis $(\epsilon_i)_{i=1}^N$ orthonormal. The simple roots are given by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i < N$, and by $\alpha_N = \epsilon_{N-1} + \epsilon_N$. The fundamental weights $\Lambda_i$ are given by $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$. We have 3 minuscule representations for $D_N$, the vector representation $V = V_{\Lambda_1}$ and the 2 spinor representations $V_{\pm} = V_{\Lambda_{N-1}}$ and $V_{\pm} = V_{\Lambda_N}$. Hence we have a basis of straight paths ending in $\pm \epsilon_i$, $1 \leq i \leq N$ for $V$, and ending in $\pm \epsilon_i$ for $V_{\pm}$ where we have an even number of minus signs for $V_{\pm}$ and an odd number of minus signs for $V_{\pm}$.

The decomposition of the tensor product $V^\otimes n$ is described in the path formalism as follows: The only path of length 1 is the one leading into the dominant weight $\epsilon_1$. Assuming we have constructed all possible paths $\ell'$ of length $n-1$, the paths of length $n$ are obtained as all possible extensions of paths $\ell' \in \mathcal{P}_{n-1}$ by adding a line segment $\pm \epsilon_i$ such that the end point of the extension is still in the closure of the dominant Weyl chamber, i.e. it satisfies $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1} \geq |\lambda_N|$.

It will also be helpful to consider the decomposition of tensor products of the vector representation of the orthogonal group $O(2N)$. In this case the simple modules are labeled by Young diagrams $\lambda$ such that $\lambda'_1 + \lambda'_2 \leq 2N$; here $\lambda'_i$ denotes the number of boxes in the $i$-th column of $\lambda$. Let $\hat{\lambda}$ be the diagram which has $2N - \lambda'_1$ boxes in the first column, and with $\hat{\lambda}'_i = \lambda'_i$ for $i > 1$. It is easy to see that if $\lambda'_1 + \lambda'_2 \leq 2N$, then also $\hat{\lambda}$ is a Young diagram. To describe the restriction rule from $O(2N)$ to $SO(2N)$, we will, as usual, identify a Young diagram with $\leq N$ rows with a vector in $\mathbb{Z}^N$. The weight $\hat{\lambda}$ is defined by $\hat{\lambda}_N = -\lambda_N$ and
\[ \lambda_i = \lambda_i \text{ for } i < N. \] Moreover, \( V^{(O)} \) and \( V^{(SO)} \) refer to \( O(2N) \) and \( SO(2N) \)-modules. Then we have

(a) If \( \lambda_i' \neq N \), then \( V^{(O)}_{\lambda} \cong V^{(SO)}_{\lambda'} \) or \( V^{(O)}_{\lambda} \cong V^{(SO)}_{\lambda} \) as an \( SO(2N) \)-module, depending on whether \( \lambda_i' < N \) or \( \lambda_i' > N \),

(b) \( V^{(O)}_{\lambda} \cong V^{(SO)}_{\lambda'} \oplus V^{(SO)}_{\lambda} \), if \( \lambda_i' = N \).

There is also a simple way of describing the tensor product rules for \( O(2N) \): here \( V_{\lambda} \otimes V \) decomposes as a direct sum of simple modules \( V_\mu \), where \( \mu \) ranges over all Young diagrams obtained by adding or subtracting a box to/from \( \lambda \).

It was shown by Brauer that the algebra \( \text{End}_{O(2N)}(V^\otimes n) \) is generated as an algebra by the elements \( 1_i \otimes a \otimes 1_{n-i-2} \), where \( a \in \text{End}_{O(2N)}(V^\otimes 2) = \text{End}_{SO(2N)}(V^\otimes 2) \) (for \( N > 1 \)), and \( 1_j \) is the identity on \( V^\otimes j \). For \( \text{End}_{SO(2N)}(V^\otimes n) \), we need an additional generator, called the Pfaffian. It is the unique \( SO(2N) \)-invariant map on the \( N \)-fold antisymmetrization of \( V \) which maps \( v_1 \land v_2 \land \ldots \land v_N \) to \( v_{N+1} \land v_{N+2} \land \ldots \land v_{2N} \), where \( (v_i) \) is an orthonormal basis for \( V \).

2. Roots, weights and Casimirs for \( E_\lambda \)

2.1. An \( E_N \) series. We consider Coxeter graphs of type \( E_N \), with \( N > 5 \) and \( N \neq 9 \), and with the vertices labeled as below.

![Coxeter graph](image)

Figure 1

We can give an explicit construction of the root system for \( E_N \) as follows. Let \( \{\epsilon_i\} \) be the standard basis for \( \mathbb{R}^N \), which we supply with a symmetric bilinear form for which \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{ij} \) for \( i \neq 0 \) and \( \langle \epsilon_0, \epsilon_0 \rangle = 1/(9 - N) \). Hence the form is positive definite for \( N < 9 \) and has signature \( (N - 1, 1) \) for \( N > 9 \). The simple roots are given by

\[
\alpha_i = \begin{cases} 
\epsilon_i - \epsilon_{i+1} & \text{if } i \leq N - 2, \\
\epsilon_i + \epsilon_{i+1} & \text{if } i = N - 1, \\
\frac{1}{2}(9 - N; -1, -1, \ldots -1, 1) & \text{if } i = 0,
\end{cases}
\]

(2.1)

Observe that removing the vertex 0 results in the diagram \( D_{N-1} \); hence \( \{\alpha_i\}_{i=1}^{N-1} \) defines a set of simple roots for type \( D_{N-1} \) which lives in the subspace of \( \mathbb{R}^N \) spanned by \( \epsilon_i, 1 \leq i \leq N - 1 \). So in notations of Section 1.1 we have \( g_0 = so_{2N-2} \). Moreover, if \( \omega \) is a weight of our root system of type \( E_N \), the corresponding weight \( \tilde{\omega} \) of \( g_0 \) is obtained by removing the 0-th
coordinate of $\omega$. We next describe the dominant weights. It is easy linear algebra to check that the equation $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ imply that $\Lambda_0 = (2; 0, \ldots, 0)$, $\Lambda_i = (i; 1, \ldots, 1, 0, \ldots, 0)$, with $i$ 1's, for $0 < i < N - 2$, $\Lambda_{N-2} = \frac{1}{2}(N - 1; 1, \ldots, 1, -1)$ and $\Lambda_{N-1} = \frac{1}{2}(N - 3; 1, \ldots, 1, 1)$.

It follows that for any weight $\lambda$ the weight $\tilde{\lambda}$ can be identified with the last $N - 1$ coordinates of the vector representing $\lambda$. We say that $\lambda$ is an integer or a half-integer weight if all the coordinates of $\tilde{\lambda}$ are integers or half-integers respectively. Writing $\lambda$ as a linear combination $\sum m_i \Lambda_i$ one sees easily that $\lambda$ is an integer weight if and only if $m_{N-2} + m_{N-1}$ is even. In the following we define for any weight $\omega$ the weight $\hat{\omega}$ by $\hat{\omega}_{N-1} = -\omega_{N-1}$ and $\hat{\omega}_i = \omega_i$ (i.e., we change the sign of the $N - 1$-coordinate).

Moreover, we define for each weight $\omega$ the number $k(\omega)$ by one of the following 2 ways:
(a) $k(\omega) = -\langle \omega, 2\alpha_0 \rangle$, where $\alpha_0 = \frac{1}{2}(0 - N - 1, -1, \ldots, -1) = \alpha_0 = \epsilon_{N-1}$,
(b) If $\omega = \sum_i m_i \Lambda_i$, then $k(\omega) = m_{N-1} - m_{N-2} - 2m_0$.

The equivalence of statements (a) and (b) is easy to show; it suffices to check that $\langle \Lambda_i; 2\alpha_0 \rangle$ is equal to 0 if $0 < i < N - 2$, and that it is equal to 2, 1 and $-1$ for $i = 0$, $N - 2$ and $N - 1$ respectively. Also observe that we have
(c) $k(\omega) = |\omega| - \omega_0$, if all entries of $\omega$ are positive, and
(d) $k(\lambda) \leq 2\lambda_{N-1}$, for any dominant weight $\lambda$.

Here statement (d) follows from $k(\lambda) = -\langle \lambda, 2\alpha_0 \rangle = -\langle \lambda, 2\alpha_0 \rangle + 2\lambda_{N-1}$.

We now define a set , which will be useful for labeling dominant weights of $E_N$ and for labeling representations of braid groups. The choice of notation is motivated by the results on fundamental modules in the following section. As before we will identify Young diagrams with dominant weights of $D_N$ without change of notation.

**Definition 2.1.** (a) The set , consists of all triples $(n; \mu, i)$ where $n \in N$, $\mu$ is a Young diagram with $|\mu| \leq n$, and $i \in \{0, 1, 2\}$; moreover, if $i = 0$, $n - |\mu|$ is even, if $i = 1$, $|\mu| \leq n - 3$, and if $i = 2$, $|\mu| = n - 6$. For given $n \in N$, the set , $(n)$ consists of all elements of , whose first coordinate is equal to $n$.

(b) The map $\Phi$ assigns to each integral dominant weight of $E_N$, $N \neq 9$, an element in , as follows:
(0) If $\lambda$ is integer and $\lambda_0 - |\lambda| \geq 0$, $\Phi(\lambda) = (\lambda_0; \tilde{\lambda}, 0)$ if $\lambda_0 - |\lambda| \geq 0$, and $\Phi(\lambda) = (\lambda_0; \tilde{\lambda}, 0)$ if $\lambda_0 - |\lambda| < 0$.

(1a) If $\lambda$ is half-integer with $k(\lambda) \leq 1$ and $\lambda_{N-1} \leq 0$, then $\Phi(\lambda) = (\tilde{\lambda} + \alpha_0, 1)$
(1b) If $\lambda$ is half-integer with $k(\lambda) \leq 1$ and $\lambda_{N-1} > 0$, then $\Phi(\lambda) = (\tilde{\lambda} + \alpha_0, 1)$,
(2) If $k(\lambda) = 2$, then $\Phi(\lambda) = (\lambda + 2\alpha_0, 2)$,
(>2) If $k(\lambda) > 2$, write $k(\lambda) = 3k_1(\lambda) + k_2(\lambda)$, with $0 \leq k_2(\lambda) \leq 2$. Then $\Phi(\lambda) = (\lambda + (k_1(\lambda) + k_2(\lambda))\alpha_0 + k_1(\lambda)\epsilon_N, k_2(\lambda))$, (i.e. the $N$-th row of the Young diagram in $\Phi(\lambda)$ has $k_1(\lambda)$ boxes).

**Remark 2.2.** Observe that the level 0 weights $\lambda$ and $\tilde{\lambda}$ get mapped to the same element in ,. As $\tilde{\lambda} \neq \lambda$ for $\lambda_{N-1} \neq 0$, $\Phi$ is not injective. This can be explained in terms of centralizer algebras and their subalgebras generated by $R$-matrices similar to the discussion in Subsection 1.5. In all other cases, $\Phi$ is injective.
Lemma 2.3. (a) A dominant weight $\lambda$ of type $E_N$ is of the form $\lambda = (\lambda_0; \tilde{\lambda})$, where $\tilde{\lambda}$ is a dominant weight of $so_{2N-2}$ such that $\langle \tilde{\lambda}, 2\epsilon_- \rangle \leq \lambda_0$.

(b) $\Phi$ does indeed map into $\mathcal{S}$. If $n < N - 1$, $(n)$ is contained in the image of $\Phi$.

(c) The image of $\Phi$ consists of triples $(n; \mu, i) \in \mathcal{S}$, for which $\mu$ has at most $N$ rows; if it has exactly $N$ rows, $|\mu| = n - 3i$.

Proof. It follows from $\langle \lambda, \alpha_i \rangle \geq 0$ for $i < N$ that $\tilde{\lambda}$ must be a dominant weight of type $D_{N-1}$. Observe that $\alpha_0$ can be written as $\alpha_0 = (\frac{1}{2}(9-N); -\epsilon_\omega)$, where $\epsilon_\omega$ is the highest weight of a spinor representation. The first statement of the claim now follows from $\langle \lambda, \alpha_0 \rangle \geq 0$.

To prove that $\Phi(\lambda)$ is indeed in $\mathcal{S}$, is now straightforward. E.g. to prove the statement about $|\mu|$ in case (1b), observe that

$$n - |\mu| = 2\langle (n; \mu), \alpha_0 \rangle = 2\langle \lambda + \alpha_0, \alpha_0 \rangle = 4 + 2\langle \lambda, \alpha_0 \rangle;$$

this means that $n - |\mu| \geq 4$ and even for $(n, \mu, 1) = \Phi(\lambda) \ (\text{in case (1b)}).$ Similarly, one shows that $n - |\mu| \geq 3$ and odd for for $(n, \mu, 1) = \Phi(\lambda)$ in case (1a). Case (2) is proved similarly.

For case (2) above observe that

$$2\mu_{N-1} = 2\lambda_{N-1} - k_1(\lambda) - k_2(\lambda) \geq k(\lambda) - k_1(\lambda) - k_2(\lambda) = 2k(\lambda) = 2\mu_N.$$

Hence $\mu$ is indeed a Young diagram. Also observe that $\mu$ has $N$ nonzero rows if and only if $k(\lambda) > 2$. The statement about $|\mu|$ in (c) now follows directly from the definition of $\Phi$.

2.2. Casimir. Let $\rho = (\rho_0; N - 2, N - 3, \ldots, 1, 0)$, where $\rho_0 = 2 + (N - 1)(N - 2)/2$, and let $\lambda \in \mathbb{R}^N$. It is easy to check that $\langle \rho, \alpha_i \rangle = 1$ for $i = 0, 2, \ldots, N - 1$. We define the scalar $c_\lambda$ for any dominant weight $\lambda$ by

$$c_\lambda = \langle \lambda + 2\rho, \lambda \rangle$$

In view of our generic labeling set we will also pair $\rho$ with vectors in $\mathbb{R}^k$, $k > N$. In this case, the component $\rho_{N+i}$ is defined to be equal to $-1-i$, for $i = 0, 1, \ldots$.

Lemma 2.4. Let $\lambda, \nu, \beta, \gamma$ be weights. Moreover, define for any weight $\omega$ the weight $\bar{\omega}$ by $\bar{\omega} = \nu - \omega - \omega$. Then we have

(a) $c_{\lambda + \beta} - c_\lambda = 2\langle \lambda + \rho, \beta \rangle + \langle \beta, \beta \rangle$, and $c_\mu - c_{\mu - \beta} = 2\langle \mu + \rho, \beta \rangle - \langle \beta, \beta \rangle$,

(b) $c_{\lambda + \beta} - c_{\lambda + \gamma} = 2\langle \lambda + \rho, \beta - \gamma \rangle + \langle \beta, \beta \rangle - \langle \gamma, \gamma \rangle$,

(c) $c_{\mu_i} + c_{\mu_s} - c_\lambda - c_\omega = 2\langle \lambda + \rho, \beta_i - \beta_s \rangle - 2\langle \beta_i, \beta_s \rangle$, where $\mu_i = \lambda + \beta_i$ and $\mu_s = \lambda + \beta_s$.

Proof. The proofs are straightforward computations.

3. Fundamental modules

3.1. Weights for low level spaces.

Lemma 3.1. (a) Let $V = V_{\Lambda}$, and let $V[i]$ be as in Section 1.1. If $\omega$ is a weight in $V[i]$, then $\bar{\omega}$ is a weight in $V_{\Lambda_i} \otimes V_{\Xi}^{\otimes i}$, and $\|\bar{\omega}\|^2 \leq 1 + i + i^2(N - 9)/4$; here $V_{\Lambda_i}$ is the vector representation and $V_{\Xi}$ is a spinor representation of $so_{2N-2}$. 
More explicitly, we have, $V[0] \cong V_{\Lambda_1}$, $V[1] \cong V_{e_+}$ and $V[2] \cong V_{\Lambda_{N-6}} \oplus V_{\Lambda_{N-8}} \oplus \ldots$, a direct sum of antisymmetrizations of the vector representation of $so_{2N-2}$.

Moreover, if $\kappa$ is a weight of level $i \geq 3$, $k(\kappa) \leq 3i - 6$

(b) Similarly, the gradation of $V_{\Lambda_0}$ in terms of $so_{2N-2}$ modules is given by $V_{\Lambda_0}[0] \cong V_0$ (the trivial representation), $V_{\Lambda_0}[1] \cong V_{e_-}$ and $V_{\Lambda_0}[2] \cong V_{\Lambda_{N-6}} \oplus V_{\Lambda_{N-8}} \oplus \ldots$, a direct sum of antisymmetrizations of the vector representation of $so_{2N-2}$.

Proof. Observe that if $\omega$ is a weight in $V[i]$, then $\omega_0 = 1 - i(9 - N)/2$. If $\omega$ is conjugate to $\Lambda_1$, and hence $\|\omega\|^2 = 1/(9 - N) + 1$, we have $\|\omega_0\|^2 = \|\omega\|^2 - \omega_0^2/(9 - N) = 1 + i + i^2(9 - N)/4$. The general case follows from $\|\omega\|^2 \leq \|\Lambda_1\|^2$ (see [13], Prop. 11.4). Moreover, by Lemma 1.2, any highest weight vector in $V[i+1]$ (with respect to $\mathfrak{g}_0 = so_{2N-2}$) has a weight of the form $\omega' - \omega_0$, with $\omega'$ a weight of $V[i]$. As $-\omega_0 = \varepsilon_-$, the highest weight for a spinor representation, the claim follows by induction on $i$.

The proofs of the remaining statements are straightforward exercises using Littelmann paths and the results in Sections 1.1, 1.2 and 1.5. (there also exist other methods). Observe that $\Lambda_1$ is the highest weight of the vector representation of $so_{2N-2}$, which is a minuscule representation. So the $so_{2N-2}$-module generated by the highest weight vector is isomorphic to the vector representation. The structure of $V[1]$ can be easily deduced from the results already proved in the previous paragraph.

To compute $V[2]$, observe that the highest weight vectors for the $\mathfrak{g}_1$-action on $V[1]$ are of the form $\omega = \frac{1}{2}(N - 7; 1, \ldots, 1, -1, \ldots, -1, 1)$ with $2j$ minus signs. An easy computation shows that $\langle \omega, a_0 \rangle = j - 1$. Hence $f_{a_0} \pi_\omega = 0$ for $j \leq 1$, while for $j > 1$, $f_{a_0} \pi_\omega$ is given by the piecewise linear path

$$f_{a_0} \pi_\omega : 0 \to \frac{1}{j - 1} \omega - a_0 \to \omega - a_0.$$

One sees that these paths have endpoints $(N - 8; 1, \ldots, 1, 0, \ldots, 0)$, with $N - 2 - 2j$ ones, and we obtain a straight line if and only if $j = 2$. Moreover, observe that the height function $h_{a_0} \omega_0$ (i.e. the pairing with $\omega_0$) gives the values $0, 1 - \frac{1}{2}, 1/2$ and $0$ at the corners of the path $f_{a_0} \pi_\omega$. Hence $e_{\omega_0}^2 f_{a_0} \pi_\omega = 0$, i.e. the paths $f_{a_0} \pi_\omega$ are highest weight paths for $\mathfrak{g}_0$. These label all highest weight vectors in $V[2]$ with respect to the action of $\mathfrak{g}_0$, by Lemma 1.2 and the remark after it.

If $\omega$ is as in the previous paragraph, one similarly concludes that $f_{a_0}^2 \pi_\omega \neq 0$ only if $j \geq 3$. We thus obtain a level 3 weight $\kappa$ for which $\kappa = \Lambda_{N-8} + \varepsilon_-$ (for $N > 8$), or $\kappa = \varepsilon_-$ (for $N = 8$; there are no level 3 weights for $N < 8$, see Cor. 3.2 below). One checks easily that $k(\kappa) = 3$, using $\kappa_0 = 1 - 3(9 - N)/2$. Moreover, for all other level 3 weights $\kappa'$, we have $k(\kappa') \leq 3$; it suffices to check this for highest weights with respect to $\mathfrak{g}_0$, which can be done using Lemma 1.2 as before. Checking this is somewhat simplified after replacing the basis vector $\varepsilon_{N-1}$ by $-\varepsilon_{N-1}$. This transforms $\omega_{N-1}$ into the familiar form of a simple root of $\mathfrak{g}(A_{N-2}) = \mathfrak{g}_{-1}$, and the Weyl group for $\mathfrak{g}_{-1}$ becomes the usual $S_{N-1}$ action on positive indices.

For weights of level $i > 3$, the claim is proved by induction on $i$, using that $k(a_0) = -3$. This finishes the proof of part (a). Part (b) is done in the same way.
Corollary 3.2. (a) \( V \cong V[0] \oplus V[1] \oplus V[2] \) for \( E_6 \) and \( E_7 \), with the dimensions being equal to 27 and 56 respectively.

(b) \( V \cong \oplus_{i=0}^{4} V[i] \) for \( E_8 \), with the dimension being equal to 248.

Proof. Recall that the dimension of the vector representation of \( so_{2N-2} \) is equal to \( 2N-2 \), and that the dimension of each of its 2 spinor representations is equal to \( 2^{N-2} \). Using this, one easily checks that the sum of the dimensions of \( V[i] \), \( 0 \leq i \leq 2 \) is equal to 27 for \( E_6 \) and 56 for \( E_7 \). One can use the Littlemann algorithm to show that \( f_{\alpha_0} \) kills every path for a weight in \( V[2] \), completing the proof for types \( E_6 \) and \( E_7 \).

For \( E_8 \), one derives similarly that \( V = \oplus_{i=0}^{5} V[i] \), with \( V[3] \cong V_- \) and \( V[4] \cong V_\lambda \) and \( V[5] = 0 \). From this one can also compute that the dimension of \( V \) is 248.

Corollary 3.3. (a) The weights of \( V \) in levels 0, 1 are of the form \( \omega = (1; \pm \epsilon_i) \) (for level 0) and \( \omega = \frac{1}{2}(N - 7; \pm 1, \pm 1, \ldots, \pm 1) \), with an even number of minus signs (for level 1). We call these weights together with level 2 weights of the form \( (N - 8; w(1, 1, 1, 0, \ldots, 0)) \), with \( N - 6 \) 1's and 5 zeros, and with \( w \in S_{N-1} \) (for level 2) straight weights. Straight weights are all conjugate to the highest weight \( \Lambda_1 \) via the Weyl group.

(b) If \( \beta, \gamma \) are straight weights, \( \langle \beta, \gamma \rangle \) is equal to \( 1/(9 - N) + 1 - i \) for some nonnegative integer \( i \), with \( i = 0 \) only if \( \beta = \gamma \) (see also [13], Prop. 11.4).

(c) If \( \beta, \gamma \) are straight weights with \( \beta - \gamma = \alpha \) a simple root and \( \langle \lambda, \alpha \rangle = 0 \), then \( \lambda + \gamma \) is not a dominant weight.

(d) If \( \beta, \gamma \) are straight weights such that \( \beta + \gamma \) is a weight in \( V_{\lambda_0} \) conjugate to its highest weight, then \( \langle \beta, \gamma \rangle = 1/(9 - N) - 1 \).

(e) If \( \beta, \gamma \) are as in (d), with \( \lambda, \nu \) dominant weights such that \( \nu - \lambda = \beta + \gamma \), then \( c_{\lambda + \beta} - (c_\lambda + c_\nu)/2 + 1/(9 - N) \) is of the form \( (2N - 3) + m \), with \( l \) and \( m \) only depending on \( \Phi_{\nu} \).

Proof. Checking that the straight weights are conjugate to \( \Lambda_1 \) is straightforward (e.g. \( \omega(0) = \frac{1}{2}(N - 7; 1, \ldots, 1) = s_0(1; \epsilon_{N-1}) \), and \( (N - 8; 1, \ldots, 1, 0, \ldots, 0) = s_0(1; (N - 7; 1, \ldots, 1) - \sum_{i=N-2}^{N-5} \epsilon_i) \); the rest of the statement follows easily using the Weyl group of type \( D_{N-1} \).

For part (b), observe that \( \langle \Lambda_1, \Lambda_1 \rangle = 1/(9 - N) + 1 \). We can always conjugate the 2 weights by an element of the Weyl group such that one of them is equal to the highest weight \( \Lambda_1 \) without changing the value of the pairing. Hence \( \langle \beta, \gamma \rangle = \langle \Lambda_1, \Lambda_1 - \kappa \rangle \), where \( \kappa = \sum_{j} c_{\alpha_j} \) is a sum of simple roots. As \( V \) is an integrable highest weight module, \( \Lambda_1 - \kappa \) is a weight in \( V \) only if \( \langle \Lambda_1, \alpha_{i_j} \rangle > 0 \) for at least one summand \( \alpha_{i_j} \) of \( \kappa \). This shows (b).

For (c), we have \( \langle \lambda + \gamma, \alpha \rangle = \langle \gamma, \beta \rangle - \langle \gamma, \gamma \rangle < 0 \), by (b). For (d) it suffices to observe that \( \|A_0\|^2 = 4/(9 - N) \) and \( \|\beta\|^2 = 1 + 1/(9 - N) = \|\gamma\|^2 \); hence \( 4/(9 - N) = \|\beta + \gamma\|^2 = \|\beta\|^2 + 2\langle \beta, \gamma \rangle + \|\gamma\|^2 \), from which one easily deduces the claim.

To prove statement (e), observe that we can write a level 1 weight \( \omega \) in the form \( \omega = -\alpha_0 + (1; \epsilon_{N-1}) - \) an even number of \( \epsilon_i \), and we can write a straight level 2 weight \( \kappa \) in the form \( \kappa = -2\alpha_0 + (1; \epsilon_{N-1}) - \sum_{j=1}^{4} \epsilon_j \). Hence \( \beta - \gamma = j\alpha_0 + \) an even number of summands \( \pm \epsilon_i \). Let \( 2l \) be the sum of the signs of the summands \( \epsilon_i \). Then one observes that \( \langle \lambda + \rho, \beta - \gamma \rangle = \quad \)
l(2N – 3) + \hat{m}, with also \hat{m} independent of N. The claim now follows from (d) and Lemma 2.4(e).

**Corollary 3.4.** The values of k(\omega) for the various types of weights of V are given as follows:

0) If \omega = (1; \pm \epsilon_i) is of level 0, then k(\omega) = 0 or –2 depending on whether the sign is positive or negative.

1) Let \omega^{(0)} = \frac{1}{2}(N – 7; 1, 1, \ldots, 1). If \omega = \omega^{(0)} – \sum_e \epsilon_{ij} with an even number 2j of summands with a minus sign, then k(\omega) = 3 – 2j.

2) If \omega is a straight level 2 weight with N – 6 1’s in \bar{\Sigma}, then k(\omega) = 2; for weights \omega of level 2 of other types, k(\omega) \leq 0.

3.2. Tensor products. Let us denote the first coordinate of \Phi(\lambda) by n(\lambda), i.e. \Phi(\lambda) = (n(\lambda); \mu, i) for suitable \mu and i. Then, using notations of Definition 2.1, we can explicitly write out n(\lambda) as

\begin{equation}
n(\lambda) = \begin{cases} 
\lambda_0 & \text{if } k(\lambda) \leq 0 \text{ and } \lambda \text{ is integer}, \\
\lambda_0 + (9 – N)/2 & \text{if } k(\lambda) \leq 1 \text{ and } \lambda \text{ half-integer}, \\
\lambda_0 + (k_1(\lambda) + k_2(\lambda))(9 – N)/2 & \text{if } k(\lambda) \geq 2.
\end{cases}
\end{equation}

This can also be written in a uniform way by defining \hat{k}(\lambda) to be equal to k(\lambda) if k(\lambda) \geq 0; otherwise we define \hat{k}(\lambda) = 1 if k(\lambda) < 0 and odd, and \hat{k}(\lambda) = 0 if k(\lambda) < 0 and even. The number \hat{k}_1(\lambda) and \hat{k}_2(\lambda) are defined accordingly such that \hat{k}(\lambda) = 3\hat{k}_1(\lambda) + \hat{k}_2(\lambda). With these conventions we get

\begin{equation}
n(\lambda) = \lambda_0 + (\hat{k}_1(\lambda) + \hat{k}_2(\lambda))(9 – N)/2.
\end{equation}

**Lemma 3.5.** Let \lambda be a dominant weight. Then V_\lambda appears in V^{\otimes n(\lambda)}.

**Proof.** It suffices to find a piecewise linear path of length n(\lambda) to \lambda, within the closure of the dominant Weyl chamber, all of whose line segments are weights of V. For integer weights \lambda with k(\lambda) \leq 0, this is easy: as \lambda_0 \geq |\lambda|, one only needs level 0 weights. If \lambda is a half-integer weight with \lambda_{N-1} > 0 and k(\lambda) \leq 1, consider \gamma = \lambda – \omega^{(0)}, where \omega^{(0)} = (N – 7; 1, 1, \ldots, 1)/2 = (1; \epsilon_{N-1} – \alpha_0). As \langle \omega^{(0)}, \alpha_0 \rangle = -1, \gamma is a dominant integer weight, with \gamma_0 = \lambda_0 – (N – 7)/2. As \hat{k}(\gamma) < k(\lambda) and integer, we obtain the upper bound \gamma_0 + 1 for n, which is as stated. If \lambda_{N-1} < 0, subtract sufficiently many times the weight (1; -\epsilon_{N-1}) from \lambda until the last coordinate becomes equal to 1/2 and apply the previous case.

Assume now that \hat{k}(\lambda) = m_{N-1} – m_{N-2} – 2m_0 \geq 2, where \lambda = \sum_i m_i \Lambda_i. Let \gamma = \lambda – \hat{k}(\lambda)\Lambda_{N-1}. Expanding \gamma as a linear combination of the fundamental weights, we compute the coefficient of \Lambda_{N-1} to be 2m_0 + m_{N-2}. As all the other coefficients remain unchanged we see that \gamma is a dominant weight. As the only half-integer fundamental weights are \Lambda_{N-2} and \Lambda_{N-1}, and the sum of their coefficients is the even number 2(m_{N-2} + m_0), it also is an integer weight. To construct a path of length n(\lambda), observe that 3\Lambda_{N-1} can be reached by a path of
length \( N \) of the form
\[
(1; 1, 0, \ldots, 0) \rightarrow (2; 1, 1, \ldots, 0) \rightarrow \ldots (N - 1; 1, \ldots, 1) \rightarrow \frac{1}{2}(3(N - 3); 3, \ldots, 3).
\]

One constructs a path to \( \Lambda_{N - 1} \) of length 3 by
\[
(1; 1, 0, \ldots, 0) \rightarrow (2; 1, 1, \ldots, 0) \rightarrow \frac{1}{2}(N - 3; 1, \ldots, 1),
\]
and, similarly, a path to \( 2\Lambda_{N - 1} \) of length 6. Combining this with a path of length \( \gamma_0 = \lceil \frac{N}{2} \rceil \) to \( \gamma \) we obtain a path of length \( \gamma N - 1 + Nk_1(\lambda) + 3k_2(\lambda) \). Using the identity \( \lambda_0 = \gamma_0 + k(\lambda)(N - 3)/2 \), we see that the length is equal to \( n(\lambda) \).

**Theorem 3.6.** \( a \) The number \( n(\lambda) \) is the smallest(largest) integer \( n \) for which \( V_\lambda \) appears in \( V^\otimes n \) for \( N < 9 \) (for \( N > 9 \)).

\( b \) All segments of a path to \( \lambda \) of length \( n(\lambda) \) are straight weights.

**Proof.** If \( V_\lambda \) appears in \( V^\otimes k \), we have a path of length \( k \), all of whose line segments are weights of \( V \). Obviously, the claim follows if we can show that \( n(\lambda + \omega) \leq (\geq)n(\lambda) + 1 \) for any weight \( \omega \) of \( V \) for \( N < 9 \) (for \( N > 9 \)). If \( \omega \) is a weight of level \( j \), \( \omega_0 = 1 - j(9 - N)/2 \). Hence we get, using Eq. 3.2
\[
n(\lambda + \omega) = \lambda_0 + 1 + (k_1(\lambda + \omega) + k_2(\lambda + \omega) - j)(9 - N)/2
\]
Using again Eq. 3.2, we obtain
\[
n(\lambda + \omega) - (n(\lambda) + 1) = (k_1(\lambda + \omega) + k_2(\lambda + \omega) - j) - k_1(\lambda) - k_2(\lambda)); (9 - N)/2.
\]
Hence the proof of part(a) is reduced to proving the equation
\[
(3.3) \quad \tilde{k}_1(\lambda + \omega) + \tilde{k}_2(\lambda + \omega) \leq \tilde{k}_1(\lambda) + \tilde{k}_2(\lambda) + j
\]
Similarly, the proof of part \( b \) follows as soon as one has shown
\[
(3.4) \quad n(\lambda + \omega) = n(\lambda) + 1 \quad \text{only if } \omega \text{ is a straight weight.}
\]

One of the less painful ways of checking the validity of Eq. 3.3 goes as follows: Let \( \omega \) be a weight of level \( j \) such that \( k(\omega) \leq 3j - 6 \). Observe that all nonstraight weights satisfy this condition, by Lemma 3.1.(a) and Cor. 3.4. If \( k(\lambda) \geq 0 \) and \( k(\lambda + \omega) \geq 0 \), we get
\[
k_1(\lambda + \omega) \leq k_1(\lambda) + j - 2. \quad \text{As } k(\lambda + \omega) \leq k(\lambda) + 2, \text{ we get the inequality in 3.3. Equality}
\]
there can hold only if \( k_2(\lambda + \omega) = k_2(\lambda) + 2 \) and \( k(\omega) = 3j - 6 \). But then \( 3k(\lambda) \) and
also \( 3k_2(\lambda + \omega) \), contradicting \( k_2(\lambda + \omega) = 2 \). If \( k(\lambda) < 0 \), but \( k(\lambda + \omega) \geq 0 \), we have
\[
k_1(\lambda + \omega) \leq k_1(\lambda) + k_2(\lambda), \quad \text{except for } k(\lambda) = -1. \quad \text{In the latter case, one shows Eq. 3.3}
\]
directly as before; in all the other cases it follows from what we have already proved before.
The cases with \( k(\lambda + \omega) < 0 \) are easy and are left to the reader. Hence whenever \( \omega \) is not straight, \( n(\lambda + \omega) \leq n(\lambda) \). This proves part \( b \), assuming part \( a \) has been shown.

Moreover, by the last paragraph of page 15, it suffices to prove Eq. 3.3 with \( \omega \) a straight weight. This can be checked in a straightforward way, using Cor 3.4. One can even go a bit further to determine necessary conditions so that equality holds in 3.3, as follows: One computes the values \( k_1(\lambda + \omega) \) and \( k_2(\lambda + \omega) \) for each case of \( k_2(\lambda) = i \in \{0, 1, 2\} \), and for each value \( k(\omega) = \ldots \)
3, 2, 1, 0, −1, −2 (these are the only possibilities left for \(k(\omega)\) with \(\omega\) straight, by the results of the previous paragraph, if \(k(\lambda) \geq 0\)). The quantity \(j = k_1(\lambda + \omega) + k_2(\lambda + \omega) - k_1(\lambda) - k_2(\lambda)\) now determines the level \(j\) at which \(\omega\) has to be so that equality holds in equation 3.3. To give an example: Let \(k_2(\lambda) = 1\), and let \(k(\omega)\) run through the values 3, 2, 1, 0, −1, −2. Then equality in 3.3 would hold only if \(j = 1, 0, 1, 0, -1, 0\). As there are no level -1 weights, we see that \(n(\lambda + \omega) = n(\lambda) + 1\) for a level 1 weight \(\omega\) only if \(k(\omega) = 3\) or \(k(\omega) = 1\), i.e. \(\omega\) would have at most 2 negative entries (see Cor. 3.4). One also sees that equality in 3.3 can not hold in this case if \(\omega\) has level 2. For \(n(\lambda + \omega) > n(\lambda) + 1\), we would have to find \(\omega\) of level \(\lambda\), with \(j\) as above. It is easily checked that this can not happen, again using Cor. 3.4. One can treat the cases \(k_1(\lambda) = 0\) and \(k_1(\lambda) = 2\) similarly, except that one has to be a little bit more careful if \(k_2(\lambda) = 0\), and \(k(\lambda) < 0\): in this case a level 1 weight \(\omega\) may have more than 4 negative entries. This finishes the proof of Theorem 3.6. Moreover, we get the following detailed information about when \(n(\lambda + \omega) = n(\lambda) + 1\):

**Case (a):** Assume that \(k(\lambda) \equiv 2 \mod 3\): Then \(n(\lambda + \omega) = n(\lambda) + 1\) only if \(\omega = (1; \epsilon_j), 1 \leq j \leq N - 2\), or if \(\omega = \omega[0]\).

**Case (b):** Let \(k(\lambda) \equiv 1 \mod 3\). Then \(n(\lambda + \omega) = n(\lambda) + 1\) only if \(\omega = (1; \pm \epsilon_j), 1 \leq j \leq N - 1\), or if \(\omega\) is a level 1 weight with 0 or 2 negative entries (for \(k(\lambda) \geq 1\)).

**Case (c):** Let \(k(\lambda) \equiv 0 \mod 3\). Then \(n(\lambda + \omega) = n(\lambda) + 1\) only if \(\omega = (1; \pm \epsilon_j), 1 \leq j \leq N - 1\), or if \(\omega\) is a level 1 weight with 0, 2 or 4 negative entries (if \(k(\lambda) > 0\)) or if \(\omega\) is a level 2 weight with \(k(\omega) = 2\). If \(k(\lambda) \leq 0\), level 1 weights may have more negative entries.

We define \(\text{lev}(\lambda) = \text{lev}_{n[\lambda]}(\lambda)\) (see Section 1.1). Using the last theorem and Eq. 3.2, we obtain

**Corollary 3.7.** \(\text{lev}(\lambda) = \text{lev}_{n[\lambda]}(\lambda) = \tilde{k}_1(\lambda) + \tilde{k}_2(\lambda)\)

**Lemma 3.8.** Let \(\lambda, \nu\) be dominant weights with \(n(\nu) = n(\lambda) + 2\) and \(\text{lev}(\nu) = \text{lev}(\lambda) + 2\). Then \(|\tilde{\nu} - \tilde{\lambda}| \geq N - 5\).

**Proof.** It follows from the definitions that \(\mu_0 = n(\mu) - \text{lev}(\mu)(9 - N)/2\), for any dominant weight \(\mu\). Hence \(\nu_0 - \lambda_0 = N - 7\). Using Eq. 3.2, we get

\[
n(\nu) - n(\lambda) = \nu_0 - \lambda_0 + (\tilde{k}_1(\nu) + \tilde{k}_2(\nu) - \tilde{k}_1(\lambda) - \tilde{k}_2(\lambda))(9 - N)/2.
\]

It follows that \(\tilde{k}_1(\nu) + \tilde{k}_2(\nu) - \tilde{k}_1(\lambda) - \tilde{k}_2(\lambda) = 2\). Simple number theory shows that this is possible only if \(k(\nu) \geq k(\lambda) + 2\). If \(\nu - \lambda = \beta + \gamma, k(\beta) + k(\gamma) \geq 2\) implies the claims, using Cor. 3.4.

**Definition 3.9.** We call \(V^{\otimes n}_{\text{new}}\) the maximum direct summand of \(V^{\otimes n}\) for which each simple direct summand has as highest weight a weight \(\lambda\) with \(n(\lambda) = n\).

**Proposition 3.10.** Assume \(N > n\). Then the branching rules for \(V^{\otimes 1}_{\text{new}} \subset V^{\otimes 2}_{\text{new}} \subset \ldots \subset V^{\otimes n}_{\text{new}}\) do not depend on \(N\).

**Proof.** First of all observe that for \(n < N\) the only dominant weights \(\lambda\) for which \(n(\lambda) = n\) are the ones for which \(k(\lambda) \leq 2\). Indeed, \(k(\lambda) \geq 3\) implies \(k_1(\lambda) \geq 1\), and hence the diagram \(\mu\) in \(\Phi(\lambda)\) would have \(N\) rows; this would imply \(n(\lambda) \geq |\mu| \geq N\). We have already seen in
Lemma 2.3, (b) and its corollary that we can use the same labeling set for the highest weights in $V_{\text{new}}^\otimes N$ for any sufficiently large $N$. It suffices to translate the branching rules into this generic setting, and to check that they do not depend on $N$ as well. This is straightforward if slightly tedious. The result is described in detail in the following corollary.

Corollary 3.11. The set $\mathcal{P}$ is a partially ordered set (poset), with the order defined as the transitive closure of $\gamma \in \lambda, (n) < \gamma' \in \lambda, (n + 1)$ if $V_{\Phi^{-1}(\gamma')} \subset V_{\Phi(\gamma)} \otimes V$ for $\mathfrak{g}(E_N)$-modules with $N$ sufficiently large. Explicitly this means

1. We have $\gamma = (n; \mu, i) < \gamma' = (n + 1; \nu, i)$ iff either $\nu = \mu$ and $i = 1$, or if $\nu$ differs from $\mu$ by a single box.
2. We have $\gamma = (n; \mu, i) < \gamma' = (n + 1; \nu, i + 1)$ iff $\nu$ is a subdiagram of $\mu$ with $\mu_i - \nu_i \leq 1$ for all $i$; moreover, if $i = 1$, $|\nu - \mu|$ must be even.
3. We have $\gamma = (n; \mu, 0) < \gamma' = (n + 1; \nu, 2)$ iff $\nu$ can be obtained from $\mu$ by subtracting 5 boxes in 5 different rows.

3.3. Regular paths. For this subsection $\lambda$ and $\nu$ will always be dominant weights with $n(\nu) = n(\lambda) + 2$. Moreover, we assume that the set $\mathcal{P}(\lambda, \nu) = \mathcal{P}_2(\lambda, \nu)$ is nonempty. It follows from Theorem 3.6, (b) that it only contains piecewise linear paths $t$ of the form $\lambda - \mu_t - \nu$, with both $\mu_t - \lambda$ and $\nu - \mu_t$ being straight weights. The motivation of the following definition will become clear in Section 4.

Let $t \in \mathcal{P}(\lambda, \nu)$. We define the quantity $e(t)$ by the equation

$$e(t) = c_{\mu_t} - c_\lambda + c_\nu / 2 + 1 / (N - g)$$

Let $\beta_t = \mu_t - \lambda$ and let $\beta_t = \nu - \mu_t$; observe that both $\beta_t$ and $\beta_t$ are weights of $V$ by definition of our paths. Then we can also express $e(t)$ by the formula

$$e(t) = \langle \lambda + \rho, \beta_t - \hat{\beta}_t \rangle - \langle \beta_t, \hat{\beta}_t \rangle + 1 / (9 - N).$$

As $\langle \beta_t, \hat{\beta}_t \rangle \equiv 1 / (9 - N) \mod \mathbb{Z}$, this also shows that $e(t) \in \mathbb{Z}$.

Definition 3.12. We call a path $s$ in $\mathcal{P}(\lambda, \nu)$, going through $\mu_s$, regular if $e(s) \neq \pm 1$ and if $e_{\mu_s} \neq e_{\mu_t}$ for any other path $t \in \mathcal{P}(\lambda, \nu)$, $t \neq s$. If $\mathcal{P}(\lambda, \nu)$ only contains one path, it will always be called regular, regardless of the value of $e(t)$.

It will be important to determine when paths are regular. We have the following results:

Lemma 3.13. Let $\lambda, \nu$ be as described at the beginning of this subsection. Then

1. $V_{\nu}$ appears in $V_{\lambda} \otimes V_{\Lambda_0}$ with multiplicity $\leq 1$.
2. If the multiplicity in (a) is zero, $\mathcal{P}(\lambda, \nu)$ only contains regular paths, and at most 2 of them, i.e. the multiplicity of $V_{\nu}$ in $V_{\lambda} \otimes V_{\Lambda_0}^\otimes 2$ is at most 2.
3. Assume $\lambda$ and $\nu$ have the same level. Then all paths in $\mathcal{P}(\lambda, \nu)$ are regular, except possibly if $\lambda$ is a level 0 weight with $\lambda_{N-1} = 0$ and $\nu = \lambda$.

Proof. We will use the explicit description of weights of low level for $V_{\Lambda_1}$ and $V_{\Lambda_0}$ in Lemma 3.1 and Corollary 3.3. Assume we can write $\nu - \lambda$ as the sum of two level 0 weights $\beta^{(1)}$ and
\( \beta^{(2)} \). Hence \( \nu - \lambda = (2; \gamma) \), where \( \gamma = \pm \epsilon_i \pm \epsilon_j \). If \( \gamma \neq 0 \), then necessarily \( \beta^{(1)} = (1; \pm \epsilon_i) \) and \( \beta^{(2)} = (1; \pm \epsilon_j) \), or vice versa. Hence we have at most 2 paths of length 2 from \( \lambda \) to \( \nu \), and \( \nu - \lambda \) is not a weight in \( V_{A_0} \).

If \( \nu - \lambda \) is the sum of a level 1 weight and a level 0 weight \( \beta \), we have \( \| \bar{\nu} - \bar{\lambda} \|_\infty = 1/2 \) or \( = 3/2 \). One checks easily in the first case that \( \nu - \lambda \) is a level 1 weight of \( V_{A_0} \), which has multiplicity 1. In the second case, there exists an index \( r \) for which \( \beta_r = \pm 1 \) and \( \gamma_r = \pm 1/2 \), with matching signs. Assuming + signs, we necessarily get \( \beta = (1; \epsilon_r) \) and \( \gamma = \nu - \lambda - \beta \), i.e., we have 2 paths at the most from \( \lambda \) to \( \nu \), and \( \nu - \lambda \) is not a weight of \( V_{A_0} \) (see Lemma 3.1.(b)).

If \( \nu - \lambda \) is the sum of 2 level 1 weights, or one level 0 and one level 2 weight, we have \( |\bar{\nu} - \bar{\lambda}| \geq N - 5 \), by Lemma 3.8. If \( \| \bar{\nu} - \bar{\lambda} \|_\infty = 2 \), then \( \nu - \lambda \) is not a weight in \( V_{A_0} \) and one shows as before that the only possible segments are a level 0 weight \( \beta = (1; \epsilon_i) \) and a level 2 weight \( \kappa = \nu - \lambda - \beta \), i.e., we have at most 2 paths from \( \lambda \) to \( \nu \). Similarly, if \( \| \bar{\nu} - \bar{\lambda} \|_1 = 1 \) and \( \| \bar{\nu} - \bar{\lambda} \|_1 \geq N - 3 \), then \( \nu - \lambda \) can only be written as a sum of 2 level 1 weights, one of them necessarily \( \omega^{(0)} \), which also determines the other weight. Again, one checks that in the remaining case \( \| \bar{\nu} - \bar{\lambda} \|_\infty = 1 \) and \( \| \bar{\nu} - \bar{\lambda} \|_1 = N - 5 \) the difference \( \nu - \lambda \) can only be a weight of \( V_{A_0} \) of multiplicity 1.

If \( \nu - \lambda \) is written as a sum of a level 2 weight and a level 1 weight, one deduces again from \( n(\mu) = n(\lambda) + 2 \) that necessarily the level 1 weight has to be equal to \( \omega^{(0)} \), and that it has to be added first to \( \lambda \), i.e., there is at most one path from \( \nu \) to \( \lambda \).

Showing that all paths in case (b) are regular is straightforward: Let \( \nu - \lambda = \beta + \gamma \), with \( \beta \), \( \gamma \) weights in \( V \). One checks easily that \( P(\lambda, \nu) \) only contains one path if \( \beta = \gamma \). If \( \beta \neq \gamma \), one checks from the analysis above that \( \langle \beta, \gamma \rangle = 1/(9 - N) \) and hence \( e(\lambda + \beta) = (\lambda + \rho, \beta - \gamma) \). Moreover, one also reads off from our analysis before that \( \beta - \gamma \) is \( \pm \) the sum of positive roots. Hence \( |e(\lambda + \beta)| \leq 1 \) only if \( \langle \lambda, \beta - \gamma \rangle = 0 \) and \( \beta - \gamma \) is equal to \( \pm \) a simple root. This is ruled out by Corollary 3.3.(c). The only remaining part, checking the case with \( \bar{\nu} = \bar{\lambda} \) in case (c), is shown in a similar way.

The following technical proposition will only be needed in Section 5 except for Example 4.8, which could also be done by more elementary methods.

**Proposition 3.14.** Assume the generic case with \( n < N \). Then there are at most 2 nonregular paths in \( P(\lambda, \nu) \) for any \( \lambda, \nu \) with \( n(\nu) = n(\lambda) + 2 \). If we have exactly 2 nonregular paths \( s \) and \( t \), then \( e(t) = e(s) = 1 \).

**Proof.** Special cases have already been proved in Lemma 3.13. It will be shown in Lemmas 3.15 and 3.16 in the next subsection that we have at most 2 nonregular paths in the remaining cases. Moreover, we have exactly 2 nonregular paths, one of them, say \( t \), of the form \( \lambda \to \lambda + \beta \to \lambda + \beta + \gamma = \nu \), the other one is obtained by interchanging \( \beta \) with \( \gamma \). Moreover, \( \nu - \lambda \) has to be a weight of \( V_{A_0} \). It follows from Corollary 3.3.(d) that \( \langle \beta, \gamma \rangle = 1/(9 - N) - 1 \), and hence \( e(t) = \langle \lambda + \rho, \beta - \gamma \rangle + 1 \). If \( s \) is the other path, \( e(s) = \langle \lambda + \rho, \gamma - \beta \rangle + 1 \). It is now easy to check that \( e(s) \) and \( e(t) \) can only have critical values at the same time if they both are equal to 1.
3.4. Lemmas for Proposition 3.14. We only need to look for possible nonregular paths in \( P(\lambda, \nu) \) if \( \nu - \lambda \) is a weight of \( V_{\lambda_0} \), by Lemma 3.13 (b). If \( \nu - \lambda = \beta + \gamma \), with \( \beta, \gamma \) straight weights, we have \( \langle \beta, \gamma \rangle = 1/(9 - N) - 1 \), by Cor. 3.3 (d). Hence if \( t \) is the path \( \lambda - \lambda + \beta - \nu \), we obtain

\[
e(t) = c_{\lambda+\beta} - c_{\lambda+\gamma} = \langle \lambda + \rho, \beta - \gamma \rangle.
\]

Lemma 3.15. Assume that \( \text{lev}(\nu) - \text{lev}(\lambda) = 1 \), and that \( n(\nu) - n(\lambda) = 2 \). Then the only possible nonregular paths in \( P(\lambda, \nu) \) have as level 0 segment \( \beta = (1; -\epsilon_{k_1}) \), where \( k_1 \) is the smallest index \( r \) for which \( \nu_r < \lambda_r \). In particular, there are at most 2 nonregular paths in \( P(\lambda, \nu) \).

Proof. Let \( \nu - \lambda = \beta + \gamma \) be a weight of \( V_{\lambda_0} \), and with \( \beta = (1; \pm \epsilon_r) \). Recall that the level 1 weight \( \gamma \) can be written as \( \gamma = -a_0 + (1; \epsilon_{N-1}) - \sum_{j=1}^{2d} \epsilon_{k_j} \). This implies that

\[
e(t) = (\lambda + \rho, \beta - \gamma) + 1 = (\lambda + \rho, a_0 \pm \epsilon_r - \epsilon_{N-1} + \sum_{j=1}^{2d} \epsilon_{k_j}).
\]

Case 1: Let \( \beta = (1; \epsilon_r) \). Then we have \( r = k_m \) for some \( m \) (as \( |\nu_r - \lambda_r| = 1/2 \)), and, in particular, \( \gamma \neq \omega^{(0)} \), i.e. \( d \geq 1 \). It follows easily from (*) that \( \beta - \gamma \) is the sum of at least 3 simple roots, except possibly if \( r = N - 1 \) and \( \gamma = \omega^{(0)} - \epsilon_{N-2} - \epsilon_{N-1} \). In the first case, we get \( |(\lambda + \rho, \beta - \gamma)| \geq 3 \), which implies \( t \) to be regular. In the 2nd case, one checks that \( |(\lambda + \rho, \beta - \gamma)| = 2 \) only if \( (\lambda, \epsilon_{N-2} + \epsilon_{N-1}) = 0 \), i.e. \( \lambda_{N-2} + \lambda_{N-1} = 0 \). One checks easily that the only possible paths \( t \) would have \( e(t) = 3 \). Hence any path in \( P(\lambda, \nu) \) for which the level 0 segment is of the form \( \beta = (1; \epsilon_r) \) is regular.

Case 2: Let \( \beta = (1; -\epsilon_r) \). If \( \nu_i < \lambda_i \) for more than one index \( i > 0 \), then \( d \geq 1 \) in (*). As \( \beta, \gamma_r < 0 \), we have \( r \neq k_j \) for all \( j \). Let \( k_1 \) be the smallest of the indices \( k_j \). If \( k_1 < r \), then \( \beta - \gamma = a_0 + (\epsilon_{k_1} - \epsilon_r) + (\epsilon_{k_2} - \epsilon_{N-1}) \) has possibly more positive roots. Hence \( \langle \lambda + \rho, \beta - \gamma \rangle \geq 3 \), which implies \( e(\lambda + \beta) \neq \pm 1 \neq e(\lambda + \gamma) \). So the only paths which may not be regular are the ones for which \( \beta = (1; -\epsilon_r) \), with \( r \) the smallest index for which \( \nu_r < \lambda_r \), and \( \gamma = \nu - \lambda - \beta \). Finally, if \( \nu_i < \lambda_i \) for only one index \( i > 0 \), a segment \( \beta = (1; -\epsilon_k) \) can only appear for \( k = r \). Hence again there are at most 2 nonregular paths, by case 1.

Lemma 3.16. Let \( \text{lev}(\nu) - \text{lev}(\lambda) = 2 = n(\nu) - n(\lambda) \). Then there exist nonregular paths only if \( \nu - \lambda = 2\omega^{(0)} - \sum_{j=1}^{4} \epsilon_{k_j} \), and the segments are of the form \( \omega^{(0)} - \epsilon_{k_1} - \epsilon_{k_4} \) resp. \( \omega_0 = \epsilon_{k_2} - \epsilon_{k_3} \). In particular, there are at most 2 nonregular paths \( s \) from \( \lambda \) to \( \nu \).

Proof. By the remarks at the beginning of this subsection we can assume \( |\nu_i - \lambda_i| \leq 1 \) for all \( i > 0 \). By Lemma 3.8 the only possible level 0 segments \( \beta \) are of the form \( \beta = (1; \epsilon_r) \). One shows as in the proof of Lemma 3.15, case 1 that \( c_{\lambda+\beta} \neq c_{\lambda+\gamma} \) for any possible level 1 weight. If \( k \) is a level 2 weight, then \( c_{\lambda+\kappa} \neq c_{\lambda+\gamma} \) can be deduced from this, using \( \gamma - \kappa = (\nu - \lambda - \kappa) - (\nu - \lambda - \gamma) \). Moreover, it is easy to show that \( |e(\lambda + \beta)| > 2 \) as well as \( |e(\lambda + \kappa)| > 2 \), with \( \beta \) and \( \kappa \) as before. Hence paths in \( P(\lambda, \nu) \) involving a level 0 and a level 2 segment are regular.
So it suffices to consider paths whose segments $\gamma$ and $\gamma'$ are both level 1 weights such that $k(\gamma) + k(\gamma') \geq 2$. Hence $\gamma + \gamma' = 2\omega^{(0)} - \sum_{j}^{2d} \epsilon_{ij}$, with $2d \leq 4$. If $d = 2$, then one shows easily that $|e(\lambda + \gamma)| \geq 3$ if $\gamma$ or $\gamma'$ is equal to $\omega^{(0)}$. So we only have to consider the case when $\gamma = \omega^{(0)} - \epsilon_{k1} - \epsilon_{k2}$ and $\gamma' = \omega^{(0)} - \epsilon_{l1} - \epsilon_{l2}$, where we have $\langle \gamma, \gamma' \rangle = -1 + 1/(9 - N)$, and $\gamma - \gamma' = \epsilon_{l1} + \epsilon_{l2} - \epsilon_{k1} - \epsilon_{k2}$. One checks that this or its negative is the sum of at least 2 positive roots, except possibly if $k_1 < l_1 < l_2 < k_2$, or if $l_1 < k_1 < k_2 < l_2$. In the first case, $\gamma - \gamma'$ is the sum of only 2 simple roots only if $k_1 + 1 = l_1$ and $k_2 + 1 = l_2$ and $\lambda_{k1} = \lambda_{l1}$ as well as $\lambda_{k2} = \lambda_{l2}$. It follows that $\lambda + \gamma'$ is not dominant, while $e(\lambda + \gamma) = 3$. This shows that all paths are regular in the first case.

For the second case observe that $\langle \gamma, \gamma' \rangle = -1 + 1/(9 - N)$. Hence we have 2 nonregular paths through $\lambda + \gamma$ and $\lambda + \gamma'$ if $\langle \lambda + \rho, \gamma - \gamma' \rangle = 0$, and only one if $|\langle \lambda + \rho, \gamma - \gamma' \rangle| = 2$. The cases with $d = 0$ or $d = 1$ are easier, and can be checked similarly.

### 4. Generators of $\text{End}_{u_q} V_{\otimes n}$

#### 4.1. Basics

We will study the subalgebra of $\text{End}_{u_q} V_{\otimes n}$ generated by the $R$-matrices. As the simple module of $\text{End}_{u_q}(V_{\otimes n})$ labeled by $\lambda \in \mathcal{P}$ has a basis labeled by $\mathcal{P}(\lambda)$, the set of paths of length $n$ ending in $\lambda$, we obtain representations of the braid group $B_n$ on this path space. The matrix by which $\sigma_i$ acts on the path space was denoted by $A_i$ (see subsection 1.3).

Recall that the $R$-matrix for $V$ has eigenvalues $\alpha q, -\alpha q^{-1}$ and $\alpha q^{3-2N}$, where $\alpha = q^{1/(9-N)}$. In order to get rid of the fractions in the exponent, we shall renormalize our representation, denoting by $A_i$ the matrix derived from the braid representation which maps $\sigma_i$ to $\alpha^{-1} R_i$. Observe that these matrices satisfy the relation

\begin{equation}
A_i - A_i^{-1} = (q - q^{-1})1 - (r - r^{-1} + q - q^{-1})P_i,
\end{equation}

where $P_i$ is the eigenprojection of $A_i$ for the eigenvalue $r^{-1} = q^{3-2N}$. In the following lemma we will consider paths $t$ such that $t(i)$ is equal to $\lambda, \mu_t$ and $\nu$ for $i = n - 1, n, n + 1$, and for fixed dominant weights $\lambda$ and $\nu$; as usual, the difference $t(i) - t(i - 1)$ is a weight in $V$. Recall the definition of $e(t)$ (see 3.5) and of $M_i = A_1 A_{-1} \ldots A_{n-1} A_2 \ldots A_i$. It follows from Corollary 1.7 that $M_{n-1}$ acts on $t$ via the scalar $\alpha^{-2n+2} q^{c_{t-1} - c_t}$, and $M_n$ acts on $t$ via the scalar $\alpha^{-2n} q^{c_{t} - c_{t-1}}$ (here $c_t$ is the Casimir for $V$).

**Lemma 4.1.** Let $A = (a_{ts})$ and $P = (p_{ts})$ be the matrices obtained from the action of $A_n$ and $P_n$ on $\mathcal{P}(\lambda, \nu)$, and let $e(t), e(s)$ be defined as in 3.5. Then

\begin{equation}
(1 - q^{e(t) + e(s)})a_{ts} = (q - q^{-1}) \delta_{ts} - (r - r^{-1} + q - q^{-1})p_{ts}.
\end{equation}

This determines all entries of $A$ in terms of entries of $P$, except for $a_{tt}$, where $t \in \mathcal{P}(\lambda, \nu)$ with $e(t) = 0$. In this case, we have

\begin{equation}
a_{tt} = r^{-1} - \sum_{s \neq t} \frac{1 - q^{e(t) + e(s)}}{r - r^{-1} + q - q^{-1}} d_{st}.
\end{equation}
Proof. As $M_n = A_n M_{n-1} A_n$, we obtain $A_n^{-1} = M_{n-1} A_n M_n^{-1}$. It follows from this and the formulas stated before that

$$A_n^{-1} = \alpha^2 q^{e_{tt}+e_{ss}-e_t-e_s} a_{tt} = q^{e(t)+e(s)} a_{tt}$$

Equation 4.2 in statement now follows from 4.1. The rest of the statement will be proved after Lemma 4.3

The following lemma essentially is the classical Jucys-Murphy approach. It was used for constructing representations of Hecke algebras of type $A$, and, in a slightly different way, for constructing representations of the symmetric group.

**Lemma 4.2.** Assume that $V_{\nu} \subseteq V_{\lambda} \otimes V^{\otimes 2}$, but $\nu - \lambda$ is not a weight in $V_{A_0}$. Then all off-diagonal entries of $A = A(\lambda, \nu)$ are non-zero.

**Proof.** Let $t$ and $s$ be the 2 paths in $P(\lambda, \nu)$, with intermediate diagrams $\mu_t$ and $\mu_s$ respectively. As $A$ can be chosen to be symmetric by Lemma 1.9, it is diagonalizable over a suitable algebraic extension of $\mathbb{Q}(q)$. If $A$ only had one eigenvalue, it would be a multiple of the identity matrix. But then $a_{tt} = a_{ss}$ would also imply $e(t) = e(s)$ and $e_{\mu_t} = e_{\mu_s}$. But this is not possible by Lemma 3.13(b).

Hence we can assume that $A$ has eigenvalues $q$ and $-q^{-1}$. Using the formula for the inverse of a $2 \times 2$ matrix and Eq. 4.3, we obtain $a_{ss}/(-1) = q^{2e(t)} a_{tt}$. Using $a_{ss} + a_{tt} = q - q^{-1}$ and solving for $a_{tt}$, we obtain

$$a_{tt} = \frac{-(q - q^{-1})q^{2e(t)}}{1 - q^{2e(t)}}.$$ 

The entry $a_{ss}$ is computed in the same way. Moreover, $\det(A) = -1$ implies $a_{tt}a_{ts} = a_{ss}a_{tt} + 1$. It is checked easily that $a_{ss}a_{tt} = -1$ only if $e(t) = \pm 1$. But this would contradict regularity of $t$, shown in Lemma 3.13.

**Lemma 4.3.** Let $r = q^{2N-3}$, with $2N - 3 \neq -1$, and let $A$ and $P$ be as in Lemma 4.1. Then the diagonal entry $d_s = p_{ss}$ is equal to 0 only if $e(s) = \pm 1$.

**Proof.** By Lemma 3.13,(a), $P$ is a rank 1 idempotent. By Lemma 1.9 we can assume $P$ to be of the form $P = vv^T$, where $v$ is an eigenvector of $A$ with eigenvalue $r^{-1}$. This implies that if $p_{ss} = 0$, then the $s$-th row and $s$-th column of $P$ is equal to 0. The same statement also holds for $A$ by equation 4.2, except possibly for the diagonal entry $a_{ss}$. Therefore $a_{ss}$ must be an eigenvalue of $A$, i.e. $a_{ss} \in \{q, -q^{-1}, q^{3-2N}\}$. If $a_{ss} = q$, Equation 4.2 would imply

$$q(1 - q^{2e(s)}) = q - q^{-1},$$

from which one easily deduces that $2e(s) = -2$. One shows similarly that $a_{ss} = -q^{-1}$ would imply $e(s) = 1$. So we only need to consider the case $a_{ss} = q^{2N-3} = r^{-1}$. But as $r^{-1}$ only has multiplicity 1, this contradicts the fact that $v_s = 0$ for its eigenvector $v$.

**Conclusion of proof of Lemma 4.1:** In order to solve for $a_{ts}$ in equation 4.2, we have to check that $e(t) + e(s) \neq 0$. Assume to the contrary for 2 paths $t \neq s$. Then equation 4.2
would imply $p_{ts} = 0$, and, as $P$ is symmetric and rank 1, also $p_{tt} = d_t = 0$ and $p_{ss} = d_s = 0$. Hence this could only happen if both $s$ and $t$ were not regular. But then it would follow from Proposition 3.14 that $e(t) + e(s) = 2$.

Hence we only have to consider the case $e(t) = 0$. Again by Proposition 3.14 this can only happen for one path $t \in \mathcal{P}(\lambda, \nu)$. But as we have already computed all other entries of $A$ in terms of the entries of $P$, we can compute $a_{it}$ by considering the $tt$-entry of the equation $AP = r^{-1}P$.

Recall that representing the braid matrices on the path space via symmetric matrices leads to square roots. We now show that these representations can already be defined over $\mathbb{Q}(q)$, and are uniquely determined by the diagonal entries of certain idempotents. To be more precise, if $t$ is a path with $\lambda = t(i - 1)$ and $\nu = t(i + 1)$, and $P_t$ acts nonzero on $W(\lambda, \nu)$, we define $d_{ti} = (P_t)_{ii}$; if $P_t$ does act as the zero matrix on $W(\lambda, \nu)$, we define $d_{ti}$ to be the corresponding diagonal entry of the eigenprojection of $A$ for the eigenvalue $q$. The following proposition is a generalization of going from the orthogonal representations of the symmetric group to the semi-orthogonal representations.

**Proposition 4.4.** The representation of the braid group $B_n$ on path space is uniquely determined by the numbers $d_{ti}, t \in \mathcal{P}_n$ and $1 \leq i < n$. In particular, the representation can be defined over $\mathbb{Q}(q)$.

**Proof.** We use the notations defined before the proposition. If $\nu - \lambda$ is not a weight of $V_\lambda$, $\mathcal{P}(\lambda, \nu)$ contains at most 2 elements. It follows from Lemma 4.2 that the 2 by 2 matrix representing $A_i$ on $W(\lambda, \nu)$ is determined by its eigenprojection $\bar{P}_i$ for the eigenvalue $q$, using the equation $A_i = (q + q^{-1})\bar{P}_i - q^{-1}1$ (which holds on $W(\lambda, \nu)$, but not in general). In the other case, $A_i$ is determined by $P_i$, see Lemma 4.1. Let $d_{ti}$ be the diagonal entry of the eigenidempotent $A_i$, as discussed before, for the path $t$. Let $c_{ti} = \sqrt{d_{ti}}$ if $d_{ti} \neq 0$, and let $c_{ti} = 1$ otherwise. Now define the diagonal matrix $C = \text{diag}(d_i)$, where $c_i = \prod c_{ti}$. It is easy to see that conjugating by $C$ changes the symmetric matrices for $P_i$ and $\bar{P}_i$ into matrices all of whose entries are in $\mathbb{Q}(q)$, and each of its nonzero entries is equal to one of its diagonal entries. This, together with the previous discussion, shows the claim.

**Proposition 4.5.** Let $s$ and $t$ be regular paths in $\mathcal{P}(\lambda, \nu)$. Then $a_{st}a_{ts} \neq 0$.

**Proof.** If $P = 0$ on $W(\lambda, \nu)$, this follows from Lemma 4.2. Otherwise, both diagonal entries $d_s$ and $d_t$ of $P$ are nonzero; as $P$ is a rank 1 idempotent, this also implies $p_{st}p_{ts} \neq 0$. The claim follows from this and equation 4.2.

### 4.2. Diagonal entries
We shall see that unfortunately not all paths are regular (see example 4.8 below). This requires us to study the diagonal entries of $A$ and $P$ in more detail. We shall see later that the equation $AP = r^{-1}P$ and the braid relations will be sufficient for explicitly computing the diagonal entries of the matrix $A$. 


We shall need the following immediate consequence of Cauchy’s identity: Let \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_n \) be formal variables. Then

\[
\det \left( \frac{1}{1 - x_iy_j} \right) = \prod_{k<l}(x_k - x_l)(y_k - y_l) \prod_{1 \leq k, l \leq n}(1 - x_ky_l)
\]

(4.4)

Lemma 4.6. Let \( P = vw^T \) for 2 vectors \( v, w \), and let \( d_s = v_s w_s \) be its \( s \)-diagonal entry. Then we obtain, for each path \( t \) for which we know that \( d_t \neq 0 \) (so, in particular, for \( t \) regular), a linear equation for the diagonal entries of \( P \) given by

\[
\sum_s \frac{r - r^{-1} + q - q^{-1}}{1 - x_t x_s} d_s = \frac{q - q^{-1}}{1 - x_t^2} - r^{-1}.
\]

The resulting system of linear equations has maximum rank. In particular, if all paths are regular, the diagonal entries of \( P \) are uniquely determined by these equations.

Proof. Let \( x_t = q^{\epsilon(t)} = \alpha q^{(e_t - e_{t-1})/2} \). Then Equation 4.2 can be rewritten as

\[
(1 - x_t x_s) d_t = \left( q - q^{-1} \right) \delta_{ts} - (r - r^{-1} + q - q^{-1}) p_{ts}.
\]

By definition, \( P \) is an eigenprojection of \( A \) with eigenvalue \( r^{-1} \). Hence \( AP = r^{-1}P \), and, as \( P = vw^T \), also \( Av = r^{-1}v \). Using this and \( p_{ts} = v_s w_t \), we obtain

\[
\sum_s \frac{1}{1 - x_t x_s} ((q - q^{-1}) \delta_{ts} - (r - r^{-1} + q - q^{-1}) v_t w_s) v_t = r^{-1} v_t.
\]

If \( t \) is regular, then \( p_{tt} \) and hence also \( v_t \) are nonzero; hence the stated equality holds. Our definition of regularity also implies that \( c_{\mu t} \neq c_{\nu t} \) for paths \( t \neq \nu \). This implies \( x_t \neq x_\nu \). It follows that the square minor of the matrix for the system of linear equations derived above whose rows and columns are labeled by paths \( t \) for which \( v_t \neq 0 \) has nonzero determinant, by Equation 4.4.

Lemma 4.7. Let \( s: \lambda \to \mu \to \nu \) be a nonregular path in \( \mathcal{P}(\lambda, \nu) \). Assume that there exists a predecessor \( \delta \) of \( \lambda \) such that \( I(\delta, \mu) \) (the set of all weights which can occur halfway in a path in \( \mathcal{P}(\lambda, \nu) \) contains exactly 2 weights, \( \lambda \) and \( \delta \), where all diagonal entries of \( A_n \), acting on \( W(\lambda, \nu) \) are known. Then we can compute the diagonal entry \( a_{ss}^{(n)} \) of \( A_n \) via the braid relations.

Proof. We also denote by \( s \) the path \( \delta \to \lambda \to \mu \to \nu \). Consider the path \( t: \delta \to \lambda \to \mu \to \nu \). Observe that \( a_{st}^{(n-1)} a_{st}^{(n-1)} \neq 0 \), by Prop. 4.5.

Let us consider the matrix entry \( (A_n A_{n-1})_{tt} = (A_{n-1} A_n)_{tt} \). By assumption, the quantities \( a_{tt}^{(n)} \) and \( a_{tt}^{(n)} a_{tt}^{(n)} \) are known for all \( i \in \mathcal{P}(\lambda, \nu) \). Hence we can compute

\[
(A_n A_{n-1})_{tt} = \sum_i a_{tt}^{(n)} a_{tt}^{(n-1)} a_{tt}^{(n)}.
\]

On the other hand,

\[
(A_{n-1} A_n A_{n-1})_{tt} = a_{tt}^{(n-1)} a_{tt}^{(n-1)} a_{tt}^{(n)} + a_{tt}^{(n-1)} a_{tt}^{(n-1)} a_{tt}^{(n-1)}.
\]
The only unknown quantity in these equations is $a_{ss}^{[n]}$.

**Example 4.8.** If $\lambda = (2m - 1; \epsilon_1)$ and $\nu = (2m + 1; \epsilon)$, then $I(\lambda, \nu)$ consists of the 4 weights $\mu^{(1)} = (2m; 0), \mu^{(2)} = \lambda + \nu^{(0)}, \mu^{(3)} = (2m; \epsilon_1 + \epsilon_2)$ and $\mu^{(4)} = (2m; \epsilon - \epsilon_{N-1})$, where $\epsilon$ is as in Subsection 1.5, and $\nu^{(0)} = ((N - 7)/2; \epsilon)$ (see Cor. 3.4,(1)). One checks directly that the paths $t_1, t_2$ through $\mu^{(1)}$ and $\mu^{(2)}$ are the only possible nonregular paths (see Lemma 3.15); a direct computation shows that $\epsilon(t_1) = m - N + 2$ and $\epsilon(t_2) = N - m$. For given $m$ there are only finitely many values of $N$ for which one of these paths actually is nonregular. Except, possibly, for these cases, we can compute the diagonal entries from the $4 \times 4$ system in Lemma 4.6 (or use Proposition 5.7) to obtain.

$$d_1 = \frac{[N - 1 - m][N + m - 2][N - 2][1]}{[N - 1 - m][2N - 4][m]x}, \quad \text{and} \quad d_2 = \frac{[N - 1 - m][2N - 3][m - 1][1]}{[N - 1 - m][N + m - 3][N - 1]x},$$

where $x = [2N - 3] + [1]$. Now observe that we also obtain well-defined solutions in the cases for which one or both paths $t_1, t_2$ are nonregular (e.g., both of them are nonregular if $N = m + 1$). Hence the corresponding matrix $A$ satisfies the braid relations also in this case. Moreover, using Lemma 4.7 with $\delta = (2m - 2; 0, ... 0)$ and $\lambda = \delta + \nu^{(0)}$, we can prove uniqueness of these solutions. This method will be studied in full generality in Section 5.

### 4.3. Centralizer Algebras

Recall that the centralizer of the action of $U_q so_{2N-2}$ on the $n$-fold tensor product of the vector representation is generated by the $R$-matrices and the $q$-analog of the Pfaffian (see subsection 1.5). By Lemma 1.1,(b), the $R$ matrices for type $E_N$ do not generate all of $\text{End}_{U_q(sl_N)}(V^\otimes n)$. We have to add, at the very least, the projection in $\text{End}_{U_q(E_N)}(V^\otimes N-1)$, whose image is isomorphic to the simple module $V_{(N-1; 1, ... 1)}$. This projection is not in the algebra generated by the $R$-matrices; it will be called the quasi-Pfaffian in this paper.

**Theorem 4.9.** Let $U = U_q g(E_N)$, with $N \neq 9$, and let $C_n = \text{End}_{U}(V_{n}^\otimes)$. Then $C_n$ is generated by the $R$-matrices and the quasi-Pfaffian, restricted to $V_{n}^\otimes$.

**Proof.** We prove the theorem by induction on $n$. Let $W_n^{(n)}$ be a simple $\mathcal{C}_n$-module labeled by the weight $\nu \in \Lambda (n)$. It follows from the branching rules that

$$W_n^{(n)} = \bigoplus_{\mu} W_\mu^{(n-1)},$$

where the summation goes over the weights $\mu \in \Lambda (n-1)$ for which $\nu - \mu$ is a straight weight. Let $\mathcal{C}_n$ be the subalgebra of $\mathcal{C}_n$ generated by the $R$-matrices and the quasi-Pfaffian. By induction assumption, $\mathcal{C}_{n-1}$ acts irreducibly on each $W_\mu^{(n-1)}$.

Let $X$ be the $\mathcal{C}_n$-submodule generated by all $W_\mu^{(n-1)}$ for which $\beta = \nu - \mu$ is a level 0 weight. It is easy to see that $X$ must be a simple $\mathcal{C}_n$-module containing all the generating $\mathcal{C}_{n-1}$-submodules: Let $\omega^1$ and $\omega^2$ be 2 level 0 weights with $\omega^1 = \pm \omega^2$, and such that $\nu - \omega^i (i = 1, 2)$ as well as $\lambda = \nu - \omega^1 - \omega^2$ is dominant. Then it follows from Lemma 4.2 that $P(\lambda, \nu)$ only has 2 elements, and that $A_{n-1}$ acts with nonzero off-diagonal elements on the
corresponding path space. Hence \( W_{\omega}^{(n-1)} \) and \( W_{\mu}^{(n-1)} \) have to be in the same \( \tilde{C}_n \)-module. One deduces from this that \( X \) is simple.

Next we show that also if \( \nu - \mu = \gamma \) is a level 1 weight, \( W_{\mu}^{(n-1)} \) must be contained in \( X \). By Proposition 4.5, this follows if we can find a predecessor \( \lambda \) of \( \nu \) and 2 regular paths \( t \) and \( s \) from \( \lambda \) to \( \nu \), with \( t \) going through \( \mu \) and with \( s \) going through a weight \( \tilde{\mu} \) with the same level as \( \nu \).

Assume there exists \( \beta = (1; \epsilon_k) \) such that \( \lambda = \mu - \beta \) is dominant. Then the path \( t \) from \( \lambda \) to \( \nu \) is regular by Lemma 3.15. For constructing \( s \) we consider 2 cases.

(a) If \( \gamma_k = 1/2 \), then \( \nu_{k+1} = \nu_k \) would also force \( \gamma_{k+1} = 1/2 \), and hence \( \mu_k = \mu_{k+1} \). But this would contradict the fact that we could subtract \( \beta \) from \( \mu \). Hence we can assume \( \tilde{\mu} = \nu - \beta \) is dominant as well for \( \gamma_k = 1/2 \).

(b) If \( \gamma_k = -1/2 \), we have at least 2 negative entries in \( \gamma \), and at least one index \( r \) for which \( \lambda_r < \nu_r \). If we take for \( r \) the smallest such index, \( \tilde{\mu} = \lambda + (1; \epsilon_r) \) is dominant.

Hence in both cases (a) and (b) the path \( s \) through \( \tilde{\mu} \) is regular. So it only remains to prove the claim for weights \( \mu \) for which \( \tilde{\mu} = 0 \) or \( m \epsilon_{-1} \) for some \( m > 0 \). In the first case, this follows from Example 4.8. In the second case, we take \( \lambda = \mu - (1; -\epsilon_{N-1}) \). If \( \nu - \mu = \gamma = \omega^{(0)} - \sum_{j=1}^{2d} \epsilon_{k_j} \), we see easily that

\[
\langle \lambda + \rho, \beta - \gamma \rangle = \langle \lambda + \rho, \alpha_0 - 2\epsilon_{N-1} + \sum_{j=1}^{2d} \epsilon_{k_j} \rangle
\]

is positive (as \( \langle \lambda + \rho, -\epsilon_{N-1} \rangle \geq 0 \)). This implies the regularity of the path, as either \( \langle \beta, \gamma \rangle + 1/(9 - N) = 1 \) (if \( \nu - \lambda \) is a weight of \( V_{\lambda_0} \)), or \( \nu - \lambda \) is not a weight of \( V_{\lambda_0} \), in which case the claim follows from Lemma 4.2.

Finally, we also have to show that submodules \( W_{\mu}^{(n-1)} \) for which \( \kappa = \nu - \lambda \) is a level 2 weight are in the same \( \tilde{C}_n \)-module as the other ones. Here it suffices to observe that any path containing as a line segment a level 2 weight is regular. Moreover, if \( \nu - \lambda = 2\omega^{(0)} - \sum_{j=1}^{2d} \epsilon_{k_j} \), then also the path through \( \lambda + \omega^{(0)} - \sum_{j=1}^{2d} \epsilon_{k_j} \) is regular. This finishes the proof.

Assume now that the trivial representation \( 1 \) appears with multiplicity 1 in \( V^{\otimes k} \). Decomposing \( V^{\otimes n} \) as a direct sum of simple \( U_q \mathfrak{g} \)-modules, we define \( V_{old}^{\otimes n} \) to be the direct sum of those simple modules which already appeared in \( V^{\otimes n-k} \). Observe that \( k = 3 \) for \( \mathfrak{g}(E_6) \) and \( k = 2 \) for \( \mathfrak{g}(E_7) \) and \( \mathfrak{g}(E_8) \). Moreover, for \( \mathfrak{g}(E_6) \) and \( \mathfrak{g}(E_7) \) \( V^{\otimes n} = V_{old}^{\otimes n} \oplus V_{new}^{\otimes n} \). The following result has already more or less appeared before (see e.g. [23]), and goes back (at least for these authors) to Jones’ basic construction.

**Proposition 4.10.** The algebra \( \text{End}_U(V_{old}^{\otimes n}) \) is generated by the restrictions of the algebra \( \text{End}_U(V_{old}^{\otimes n-1}) \otimes 1 \) and of \( 1_{n-k} \otimes p \) to \( V_{old}^{\otimes n} \), where \( p \) projects onto \( 1 \subset V^{\otimes k} \).

**Proof.** The proof follows a similar pattern as the one of the last theorem. Let \( W^{(n)}_{\mu} \) be a simple \( C_n \)-module. It decomposes as a \( C_{n-1} \)-module into the direct sum of simple \( C_{n-1} \)-modules

\[
W^{(n)}_{\mu} = \oplus_{\lambda} W^{(n-1)}_{\lambda}.
\]
The claim is shown if we can find for each summand µ a nonzero idempotent \( p_\mu \in C_{n-1,\mu} \) and elements \( u = u_\mu \) and \( v = v_\mu \) in \( \tilde{C}_n \), the algebra generated by \( C_{n-1} \) and \( 1_{n-k} \otimes p \) such that \( uv = p_\mu \) and \( vu = (1_{n-k} \otimes p) \); here the equation is to be understood as an equality of the linear operators representing the algebra elements on \( W_p^{(n)} \) (it will not be true as equality of elements in the algebra). Indeed, as each \( W_p^{(n-1)} \) is a simple \( C_{n-1} \)-module, one can easily deduce that they all have to be in the same \( \tilde{C}_n \)-submodule of \( W_p^{(n)} \).

Let \( V^* \) be the dual of \( V \) (in the sense of braided tensor categories, see e.g. [14]; more details concerning this proof can be found in [23]). Moreover, let \( \tilde{p} \) be the unique \( U_q g \)-invariant projection onto \( V^* \otimes V \).

Assume \( V_\lambda \subset V^{\otimes n-k} \), and assume that \( V_\mu \subset V^{\otimes n-1} \) such that \( V_\mu \otimes V \) again contains a submodule isomorphic to \( V_\lambda \). As \( \dim \text{Hom}(V_\lambda, V_\mu \otimes V) = \dim \text{Hom}(V_\lambda \otimes V^*, V_\mu) \neq 0 \) by Frobenius reciprocity, we also have a submodule in \( V_\lambda \otimes V^* \) which is isomorphic to \( V_\mu \). Let \( p_\mu \) be the projection onto it. Then it follows that

\[
(1_\lambda \otimes p)(p_\mu \otimes 1)(1_\lambda \otimes p) = \frac{\dim_q(V_\mu)}{\dim_q V_\lambda \dim_q V}(1_\lambda \otimes p); \quad (*)
\]

(see e.g. [23], Prop. 1.4). As \( V^* \subset V^{\otimes k-1} \), we can embed \( \tilde{p} \) and \( p_\mu \) as a projection in \( \text{End}(V^{\otimes k}) \) and \( \text{End}(V_\lambda \otimes V^{\otimes k-1}) \) respectively. We get the desired elements by setting \( u = (1_\lambda \otimes p)(p_\mu \otimes 1) \) and \( v = \alpha(p_\mu \otimes 1)(1_\lambda \otimes p) \), where \( \alpha \) is the reciprocal of the fraction in \((*)\).

**Theorem 4.11.** Let \( U = U_q g(c_N) \) with \( N = 6,7 \). Then \( \text{End}_U(V^{\otimes n}) \) is generated by the \( R \)-matrices and an additional element in \( \text{End}_U(V^{\otimes N-1}) \), the quasi-Pfaffian.

**Proof.** It follows from the tensor product rules that for \( N = 6,7 \) we have \( V^{\otimes n} = V^{\otimes n}_{\text{new}} \oplus V^{\otimes n}_{\text{old}} \). Moreover, the idempotent \( p \) in the proof of Proposition 4.10 is in the algebra generated by the \( R \)-matrices: for \( N = 7 \), it is an eigenprojection of the \( R \)-matrix, while for \( N = 6 \) it is an eigenprojection of \( (R_1 R_2)^3 \) (the eigenvalues can be easily computed using Drinfeld’s quantum Casimir, see Prop. 1.6 and Cor. 1.7). Hence the claim follows by induction on \( n \) from Proposition 4.10 and Theorem 4.9.

**5. Computation of braid matrices**

**5.1. Uniqueness.** We will have the following assumptions throughout this subsection: \( \lambda \) and \( \nu \) will be dominant weights with \( n(\nu) = n(\lambda) + 2 \), \( A = A(\lambda, \nu) \) is the matrix via which the braid generator \( \sigma_{n(\lambda)+1} \) acts on the path basis labeled by \( P(\lambda, \nu) \). Moreover, we assume the generic case \( N > n \). We will show that even if we have nonregular paths in \( P(\lambda, \nu) \), \( A \) will be uniquely determined (up to transformations coming from rescaling the basis vectors, see Prop. 4.4). To do so, it suffices to check the conditions of Lemma 4.7. Recall that the block \( A(\lambda, \nu) \) is already uniquely determined if \( \text{lev}(\nu) = \text{lev}(\lambda) \), by Lemma 3.13(c) and Lemma 4.6.

**Lemma 5.1.** Assume \( \text{lev}(\nu) - \text{lev}(\lambda) = 1 \). Then \( A(\lambda, \nu) \) is unique.
Proof. It suffices to consider the case when $\nu - \lambda$ is a weight in $V_{\lambda_0}$, i.e., we can write it as

$$\nu - \lambda = \omega^{(0)} - \sum_{j=1}^{2m+1} \epsilon_j,$$

where we assume $i_1 < i_2 < \ldots < i_{2m+1}$. By Lemma 3.15, we have at most 2 nonregular paths in $P(\lambda, \nu)$, with intermediate diagrams $\mu = \lambda + (1; -\epsilon_i)$ or $\mu = \nu - (1; -\epsilon_i)$.

a) Assume that there exists an index $r \not\in \{i_j, 1 \leq j \leq 2m+1\}$ such that $\delta = \lambda - (1; \epsilon_r)$ is dominant. Let $\mu = \nu - (1; -\epsilon_i) \in I(\lambda, \nu)$. Then $\mu - \delta$ is not a weight in $V_{\lambda_0}$, and $I(\delta, \mu) = \{\lambda, \bar{\lambda}\}$, where $\bar{\lambda} = \mu - (1; \epsilon_r)$. But as $\bar{\lambda}$ has the same level as $\mu$, $A(\bar{\lambda}, \nu)$ is known.

If we only have one critical path, we can compute $A(\lambda, \nu)$ by Prop. 4.7, (b).

If we have 2 critical paths, we need a second equation, which can involve a regular path. It can be constructed as follows: Let $\delta = \lambda - (1; \epsilon_r)$, and let $\mu = \lambda + (1; -\epsilon_{2m+1})$. Then $I(\delta, \mu) = \{\lambda, \bar{\lambda}\}$, with $\bar{\lambda} = \lambda - \epsilon_r - \epsilon_{2m+1}$. As $\nu_r - \bar{\nu} = 3/2$, and hence $\lambda - \nu$ not a weight of $V_{\lambda_0}$, $A(\bar{\lambda}, \nu)$ is known, by Lemma 3.13, (a) and Lemma 4.6. This gives us the equation for another diagonal entry. This finishes the proof, in this case, by Lemma 4.7.

b) It remains to consider the case where we can not find an index $r$ as in (a). Our proof goes by induction on $m$, with $m$ as in (*). For $m = 0$, our assumption implies that $\delta = \lambda - (1; \epsilon_i)$ is dominant only if $l = i_1$. If $\lambda$ is level 0 and $n < N$, this is possible only if $\lambda = (n; k, \ldots, 0, \ldots, 0)$, and $l$ is the largest index for which we have a nonzero entry. Take $\delta = \lambda - (1; \epsilon_i)$ and $\mu = \lambda + (1; -\epsilon_{2m+1})$. Then $I(\delta, \mu) = \{\lambda, \bar{\lambda}\}$, where $\bar{\lambda} = \lambda + \epsilon_1 - \epsilon_i$. If $k > 1$ or $l > 2$, case (a) can be applied to show that $A(\bar{\lambda}, \nu)$ is uniquely determined; we will deal with the remaining cases below. To get a second equation in case we have 2 critical paths, take $\delta$ as before, and take $\mu = \lambda + \omega^{(0)}$; $\mu$ is dominant as otherwise we would only have at most one nonregular path. Then $I(\delta, \mu) = \{\lambda, \bar{\lambda}\}$, with $\bar{\lambda} = \delta + \omega^{(0)} = \nu$. As $\bar{\lambda}$ has the same level as $\nu$, $A(\bar{\lambda}, \nu)$ is uniquely determined; this gives us a diagonal entry for a nonregular path, by Lemma 4.7.

To finish the case $m = 0$, observe that we have already studied the case $\lambda = (n; 1, 0, \ldots, 0)$ in Example 4.8. Hence it suffices to consider $\lambda = (n; 1, 1, 0, \ldots, 0)$ and $\lambda = (n; 2, 0, \ldots, 0)$, with $\nu = (n+1; 2, 1, 0, \ldots, 0) + \omega^{(0)}$. Observe that these are the only 2 intermediate diagrams for $\delta = (n+1; 1, 0, \ldots, 0)$ and $\mu = (n+1; 2, 1, 0, \ldots, 0)$. Both cases, we do get an equation for the diagonal path through $\lambda + \omega^{(0)}$ as in the second equation of the last paragraph. Hence it suffices to show that for given $n$ we can have 2 nonregular paths only for one of these 2 diagrams. If $t$ is the path through $\mu$ to $\nu$, it is a straightforward computation to check that $e(t) = n/2 - N + 2$ for $\lambda = (n; 1, 1, 0, \ldots, 0)$ and $e(t) = n/2 - N$ for $\lambda = (n; 2, 0, \ldots, 0)$. This finishes the case $m = 0$ with $lev(\lambda) = 0$. If $lev(\lambda) = 1$ and $lev(\nu) = 2$, we can always assume $\lambda_{N-1} = -1/2$ and $\nu_{N-1} = 1$. If $\lambda_{N-1} = -1/2$, $P(\lambda, \nu)$ contains at most 2 paths, which are both regular (see Lemma 3.13, (b)). Otherwise, after subtracting $\omega^{(0)}$ from both $\lambda$ and $\nu$, we are essentially in the case just treated. We leave the details to the reader.

For $m > 0$, assume first that we can find 2 indices $p$ and $l$ such that both $\lambda - \epsilon_p$ and $\lambda - \epsilon_l$ are dominant. Set $\delta = \lambda - (1; \epsilon_p)$ and $\mu = \lambda + (1; -\epsilon_l)$. Then $I(\delta, \mu) = \{\lambda, \bar{\lambda}\}$, where $\bar{\lambda} = \lambda - \epsilon_p - \epsilon_l$. Now by induction assumption $A(\bar{\lambda}, \nu)$ is known so the conditions for Prop.
4.7 are satisfied. Moreover, the indices $p$ and $l$ are interchangeable, i.e., we get 2 paths which yield equations. If we have 2 critical paths, we can choose $p = i_1$, i.e., one of the paths will be critical. So we get the claim from Prop. 4.7. Finally, if all removable boxes are in the same column, then we can only have one singular path at the most, where one adds the level 1 weight first.

**Lemma 5.2.** Let $\lambda, \nu$ be dominant weights with $n(\lambda) + 2 = n(\nu)$ and with $\text{lev}(\nu) - \text{lev}(\lambda) = 2$. Then $A(\lambda, \nu)$ is uniquely determined.

**Proof.** By Lemma 3.16 we only have to consider the case where $\nu = \lambda + 2\omega^0 - \sum_{j=1}^{4} \epsilon_{k_j}$. Let $\omega^{(1)} = \omega^{(0)} - \epsilon_{k_1} - \epsilon_{k_4}$ and $\omega^{(2)} = \omega^{(0)} - \epsilon_{k_2} - \epsilon_{k_3}$. Let us first show that we can also assume that both $\lambda + \omega^{(1)}$ and $\lambda + \omega^{(2)}$ are dominant. Indeed, $\lambda + \omega^{(2)}$ not being dominant and $\lambda + \omega^{(1)} + \omega^{(2)}$ being dominant is possible only if $\lambda_{k_3} = \lambda_{k_4}$ and $k_4 = k_3 + 1$. On the other hand, $\lambda + \omega^{(1)}$ being dominant implies that $(\lambda + \rho, \epsilon_{k_1} - \epsilon_{k_2}) > 1$. As moreover $-\langle \omega^{(1)}, \omega^{(2)} \rangle + 1/\langle 9 - N \rangle = 1$ (see proof of Lemma 3.16), we see that $\epsilon(\lambda + \omega^{(1)}) > 1$, i.e., the path through $\lambda + \omega^{(1)}$ is not singular. This proves our assertion.

If both $\lambda + \omega^{(1)}$ and $\lambda + \omega^{(2)}$ are dominant, then so is $\delta = \lambda - (1; \epsilon_{k_1})$ and $\mu = \lambda + \omega^{(2)}$. As $\mu_{k_1} - \delta_{k_1} = 3/2$, we have $I(\delta, \mu) = \{\lambda, \tilde{\lambda}\}$, with $\tilde{\lambda} = \delta + \omega^{(2)}$. By Lemma 5.1, the matrix block $A(\lambda, \nu)$ is well-defined, as $\text{lev}(\nu) - \text{lev}(\lambda) = 1$. Hence we obtain an additional equation for $d_s$, where $s$ is the path going through $\mu = \lambda + \omega^{(2)}$. We obtain a second equation, for the other singular path (if it exists) by setting $\delta = \lambda - (1; \epsilon_{k_3})$ and $\mu = \lambda + \omega^{(1)}$, by the same reasoning as before. This finishes the proof of this lemma.

**Proposition 5.3.** Let $n < N$. Then the matrices $A_i$, $1 \leq i \leq n$ by which the generators $\sigma_i$ of the braid group $B_n$ act on the path basis are uniquely determined by the equations in Lemmas 4.6 and 4.7.

**Proof.** It was shown in Lemma 4.6 that we get a system of linear equations of maximum rank labeled by regular paths. If there are nonregular paths, we have checked in Lemmas 5.1 and 5.2 that we can find sufficiently many additional linear equations to obtain a nondegenerate system of linear equations for the diagonal entries of the matrix $P_i$. This determines the matrices $A_i$ by Lemmas 4.1 and Proposition 4.4, up to rescaling of the basis vectors.

5.2. **Generic braid representations.** Recall the definition of the labeling set $\mathcal{L}$, in Definition 2.1, with its partial order defined in Cor. 3.11. Also recall that the center of the braid group $B_n$ is generated by $\zeta_n = (\sigma_1, \sigma_2, \ldots, \sigma_{n-1})^n$; geometrically, it can be viewed as twisting $n$ strands by $360^\circ$. With this picture in mind, it is not difficult to check that $\zeta_{n+1}\zeta_n^{-1} = \mu_n = \sigma_n, \sigma_{n-1}, \ldots, \sigma_1$.

**Theorem 5.4.** (a) For each $\gamma = (n; \mu, i) \in \mathcal{L}$, there exists a representation $\pi_\gamma$ over the field $\mathbb{Q}(r, q)$ of rational functions in 2 variables uniquely determined by

1. $\pi_{[2, [1], 0]}(\sigma_1) = -q^{-1}, \pi_{[2, [2], 0]}(\sigma_1) = q$ and $\pi_{[2, [0], 0]}(\sigma_1) = r^{-1},$
2. $\pi_(\zeta_n)$ is a scalar, which is equal to 1 whenever $r = q^n$, $n \in \mathbb{N}$ sufficiently large, and $q \to 1$. 

(3) $\begin{pmatrix} \pi_\gamma \end{pmatrix}_{B_{n-1}} \cong \oplus_{r' < \gamma} \pi_{r'}$, where $r' \in (n-1)$.

(b) Each representation $\pi_\gamma$ is irreducible.

(c) There exists a finite-dimensional semisimple algebra $I_n$, defined over $\mathbb{Q}(r,q)$ which is isomorphic to $\oplus_{\gamma \in \Gamma(n)} \pi_\gamma(\mathbb{Q}(r,q)B_n)$

Proof. The claim will be shown by induction on $n$. Existence and uniqueness is clear for $n = 2$, by (1). For the induction step, let us first prove the uniqueness of these representations. By assumption (2), the element $\zeta_{n+1}$ acts via a scalar $\beta$ on $W_\gamma$, $\gamma \in (n+1)$. A simple comparison of determinants shows that $\beta^{\dim \gamma} = \det(\pi_\gamma(\sigma_1))^{n(n+1)}$, where $\dim \gamma$ is the dimension of $W_\gamma$. As the right hand side is a monomial in $r$ and $q$ (here monomials are allowed to have negative exponents), while $\beta \in \mathbb{Q}(r,q)$, we see that $\beta$ itself is a monomial in $r$ and $q$ with coefficient 1, by (2). Hence also the scalars via which $M_n$ and $M_{n+1}$ act on a path $t$ are monomials in $r$ and $q$. So we obtain equations for the diagonal entries of $P_1$ as in Lemma 4.1, with $q^{c[i]+c[k]}$ replaced by a monomial in $r$ and $q$. One can use the same arguments as in Proposition 5.3 and its preceding lemmas to show that the matrices for the braid generators are uniquely determined. This proves the uniqueness statement.

To prove the existence, we know that for $N > n$ the $R$-matrices for $U_q\mathfrak{g}(E_N)$, restricted to $V^{\otimes n}_{\text{ave}}$ give us braid representations which satisfy conditions (1)-(3) for $r = q^{2N-3}$. As this holds, in particular, for infinitely many values of $N$, the braid relations also must hold for the generic matrices compute above. This shows the existence for generic $r$.

5.3. The linear equation. In the following, $x_1, x_2, \ldots x_n$ are variables. The matrix $C$ is defined by $c_{ij} = 1/(1 - x_i x_j)$. The goal of this subsection is to compute the solution of the system of equations given in Lemma 4.6, assuming all paths are regular.

Lemma 5.5. Let $c = (1, 1, \ldots, 1)^T$. Then the linear equation $Cy = c$ has the solution

$$y_i = (1 - x_i^2) \prod_{j \neq i} \frac{-x_j(1 - x_i x_j)}{x_i - x_j}.$$ 

Proof. We will use Cramer's rule for solving this linear system. It follows from Cauchy's identity 4.4 that

$$(5.1) \quad \det(C) = \frac{\prod_{1 \leq k \leq n} (x_k - x_i)^2}{\prod_{1 \leq k, l \leq n} (1 - x_k x_l)}.$$ 

By symmetry, it suffices to compute the solution $y_1 = \det(D)/\det(C)$, where $D$ coincides with $C$ except for the first column, where $d_{i1} = 1$ for $i = 1, 2, \ldots n$. It suffices to observe that $D$ can be computed via Equ. 4.4 after setting $y_1 = 0$ and $y_j = x_j$ for $j > 1$. One obtains

$$\det(D) = \frac{\prod_{j=2}^n (-x_j)(x_1 - x_j) \prod_{2 \leq k \leq n} (x_k - x_i)^2}{\prod_{j=2}^n (1 - x_1 x_j) \prod_{2 \leq k \leq n} (1 - x_k x_l)}.$$ 

The statement now follows easily from Cramer's rule.
Lemma 5.6. Let the vector $b$ be given by $b_i = 1/(1 - x_i^2)$. Then the equation $Cy = b$ has the solution given by

$$y_i = x_i^\varepsilon \prod_{j \neq i} \frac{1 - x_ix_j}{x_i - x_j},$$

where $\varepsilon = 1$ for $n$ even and $\varepsilon = 0$ for $n$ odd.

Proof. We will use Cramer’s rule for solving this linear system as in Lemma 5.5. Again, using symmetry, it suffices to compute the solution $y_1 = \det(D)/\det(C)$, where $D$ coincides with $C$ except for the first column, where now $d_{11} = 1/(1 - x_1^2)$ for $i = 1, 2, \ldots, n$. The computation of $\det(D)$ is done in several steps:

Step 1: Let $\Delta_{[r,s]} = \prod_{i < k \leq s}(x_i - x_k)$. Then $\Delta_{[1,n]}\Delta_{[2,n]}$ divides $\det(D)$.

To prove this, observe that after subtracting the $j$-th row, the $i$-th row of $D$ is equal to

$$\left(\begin{array}{c}
\frac{x_i^2 - x_j^2}{(1 - x_i^2)(1 - x_j^2)} \\
\vdots \\
\frac{x_i(x_i - x_j)}{(1 - x_i x_j)(1 - x_i x_j)} \\
\vdots
\end{array}\right),$$

Hence we can factor $(x_i - x_j)$ from the determinant. Moreover, if $2 \leq i < j$, we get a second factor $(x_i - x_j)$ in the $i$-th column after subtracting the $j$-th column from it.

Step 2: Now expand $D$ with respect to the first column, and write the $C_{11}$ minor as matrix

$$z_i = \begin{cases} x_i & \text{if } l > i, \\ x_{i-1} & \text{if } l \leq i. \end{cases}$$

Then we deduce from Eqn. 4.4 that

$$\det(D) = \sum_{i=1}^n (-1)^{i+1} \frac{1}{1 - x_i^2} \prod_{2 \leq k < l \leq n} (x_k - x_l)(z_k - z_l) = \prod_{2 \leq k < l \leq n} (1 - x_k x_l) \sum_{i=1}^n (-1)^{i+1} \frac{1 - x_i^2}{1 - x_i^2} \prod_{j=2}^n (1 - x_i x_j) \prod_{1 \leq k, l \leq n (1 - x_k x_l)}.$$
Hence it su ces to consider the monomials of degree $n(n - 1)/2$, it follows from the previous sentence that $b = 0$ for $n$ even and $a = 0$ for $n$ odd, i.e. we have

\begin{equation}
F = \begin{cases} 
ax_1 \Delta[n, n] & \text{if } n \text{ even,} \\
 b \Delta[n, 1] & \text{if } n \text{ odd.}
\end{cases}
\end{equation}

Step 4: It follows from the explicit formulas for $\det(C)$ (see 4.4) and $\det(D)$ (see the last the steps) that $y_1 = \det(D)/\det(C)$ is as claimed, provided we can show that the constants $a$ and $b$ in the equation 5.3 are equal to 1. To prove this for $n$ even, observe that the coefficient $F_n$ of $x^3_i$ in $F$ is a homogeneous polynomial in the variables $x_2, \ldots, x_n$ of degree $(n - 1)(n - 2)/2$. Hence it su ces to consider the monomials of degree $n - 2$ in

$$\sum_{i=2}^{n} (-1)^{i+1} \frac{1}{1 - x_i^2} \prod_{j=2, j \neq i}^{n} (1 - x_i x_j) = \sum_{i=2}^{n} (-1)^{i+1} \prod_{j=2, j \neq i}^{n} (1 - x_i x_j).$$

These monomials are necessarily of the form $x_i^{n - 2} \cdot E_{n/2-1}(x_2, \ldots, x_n)$, where $E_k$ is the $k$-th elementary symmetric polynomial. It follows from Weyl’s character formula for type $A$ that

$$E_{n/2-1}(x_2, \ldots, x_n) \prod_{l=1}^{s} (x_k - x_l) = \det(x_r^{l_s}),$$

where the determinant is taken over the $(n - 2) \times (n - 2)$ matrix with $2 \leq r \leq n$, $r \neq i$, and $1 \leq s \leq n - 2$ with $l_s = n - 2 - s$ for $s < n/2$ and $l_s = n - 3 - s$ for $s \geq n/2$. It follows from this and elementary rules for determinants that

$$F_n = \sum_{i=2}^{n} (-1)^{i+1} x_i^{n/2} \det(x_r^{l_s}) = \det(x_p^{n-2}),$$

where now $2 \leq p, q \leq n$. This shows that $a = 1$. One shows by the same method also that $b = 1$. Having computed $E$ and hence also $\det(D)$, we get the claim using Cramer’s rule.

**Proposition 5.7.** Let $x_i = q^{e_i}$, with $e_i$ formal variables for $1 \leq i \leq n$, and let $[k] = q^k - q^{-k}$. Then, with notations of Lemmas 5.6 and 5.5, and with setting $r = q^{2N-3}$, the linear system

$$((2N - 3) + [1])Cy = (q - q^{-1})b - r^{-1}c$$

has the solution

$$y_s = \frac{[e_s - 1]}{[2N - 3] + [1]} \prod_{i \neq s} \frac{[(e_s + e_i)/2]}{[(e_s - e_i)/2]} \quad \text{if } n \text{ even},$$

$$y_s = \frac{[e_s] + [1]}{[2N - 3] + [1]} \prod_{i \neq s} \frac{[(e_s + e_i)/2]}{[(e_s - e_i)/2]} \quad \text{if } n \text{ odd.}$$

In particular, after restricting the equations of Lemma 4.6 to paths $t$ for which $d_i \neq 0$ and setting $e_i = e(t)$, we obtain for $y_s$ the nonzero diagonal entries $d_s$. 
Proof. Using Lemma 5.6 and Lemma 5.5, we obtain

\[ d_s = \frac{1}{[2N-3]+1}[q-q^{-1})x_s^r - q^{-1}(x_s^{-1})\prod_{i=1}^{n} x_i \prod_{i \neq s} 1 - x_i x_s. \]

As \( d_s \) is the diagonal entry of a \( s \)-invariant idempotent, we have \( d_s = d_s \), by Lemma 1.9(b).

As the \( x_i \)'s and \( r \) are powers of \( q \), we also have \( x_i = x_i^{-1} \) and \( r = r^{-1} \). It is straightforward to check that \( d_s = d_s \) implies

\[ \prod_{i=1}^{n} x_i = \begin{cases} rq & \text{if } n \text{ even}, \\ r & \text{if } n \text{ odd}. \end{cases} \]

\[ (5.4) \]

**Theorem 5.8.** The statements of Theorem 4.9 and Theorem 4.11 also hold for \( q \in \mathbb{C} \), \( q \) not a root of unity. In the classical case \( q = 1 \), \( C_n \) is generated by \( C_2 \), the quasi-Pfaffian, and the symmetric group \( S_n \).

Proof. As the diagonal entries \( d_s \) are quotients of products of \( q \)-integers (with possibly \( q = q^{-1} \) added), they are zero for \( q = 1 \) only if they are already zero as functions in \( q \). Hence the arguments in the proofs for the above stated corollaries and theorem also go through for \( q = 1 \) and for \( q \) not a root of unity.

The results above can also be used to compute the diagonal entries, and hence the braid matrices over \( \mathbb{Q}(r, q) \). More generally, we have

**Theorem 5.9.** Let now \( r, q \) be elements defining a field extension of \( \mathbb{Q} \). The representations of the braid group \( B_n \) into the algebra \( \mathfrak{l}_n(r, q) \) are well-defined and semisimple if \( q \) is not a root of unity, and \( r^\prime q^m \neq \pm 1 \) for \( l, k \in \mathbb{Z} \).

Proof. There is obviously no problem with substituting elements of a field extension for the variable \( r \) and \( q \) for diagonal entries \( d_s = 0 \), as rational functions. The other diagonal entries can be computed by the equations in Lemma 4.6, only running over paths \( s \) for which \( d_s \neq 0 \). One can explicitly compute from Corollary 3.3(d), that there exist integers \( l \) and \( m \) such that \( r^l q^{m} = q^{e(t)} \) for \( r = q^{2N-3} \), where \( e(t) \) was computed for \( q(E_N) \). Hence we can compute the matrix entries in terms of rational functions of \( r \) and \( q \) which are quotients of products of factors of the form \( r^l q^{m} - r^{-l} q^{-m} \), with possibly \( q = q^{-1} \) added. It is easy to check for both cases that these quantities are zero only if \( r^\prime q^m \neq \pm 1 \) for suitable integers \( l \) and \( m \). Hence our representations are well-defined and semisimple whenever \( r^\prime q^m \neq \pm 1 \).

**Remark 5.10.** It is easy to check that for \( \gamma = (\nu, \mu, 0) \) the braid representation \( \pi_\gamma \) determines the direct summand of the \( q \)-version of Brauer’s centralizer algebra, labeled by \( \mu \), which was defined in [1] and in [21]. For other elements in \( \gamma \), one does get new braid representations. For the \( q \)-Brauer algebra one also obtains interesting braid representations when it is not semisimple. Naively speaking, this can be done by just removing paths for which the diagonal entry of one of the eigenprojections of a braid generator becomes 0 (see Prop. 4.4). It would be interesting to study similar braid representations in connection with the algebra \( \mathfrak{l}_n \). Observe
that one does obtain such representations for the $E_N$-specializations, i.e. with $r = q^{2N-3}$. Results for the $q$-Brauer algebra and Hecke algebras also suggest further interesting semisimple quotients of $I_n$ for $q$ a root of unity. As one can compute the quantities $l_i$ and $m_i$ in the proof of Theorem 5.9 explicitly (see e.g. Corollary 3.3(e)), it should also be possible to determine such semisimple quotients. Similarly, it should be possible to find much stronger conditions under which $I_n$ may not be semisimple, for given $n$.

Remark 5.11. A definition of the algebra $I_n$ via generators and relations seems to be rather complicated. The relations already for $I_3$ are rather intricate, and certainly do not suffice to define $I_n$. Another possibility of defining the algebras $I_n$ would be via a given basis of reduced words (where ‘reduced’ would have to be defined appropriately). One way might be to use canonical bases, as indicated by Lusztig in [20], 27.3.10. An explicit integral basis over a suitable ring has been constructed for the $q$-Brauer algebra in a paper by Morton and Wassermann [22].

A possible reason why $I_n$ is difficult to define via generators and relations is the fact that there exist larger quotients than $I_n$ of the group algebra of the infinite braid group whose restriction to $B_2$ would only give a 3-dimensional algebra. For one, observe that one can obtain additional irreducible representations of $B_n$ after interchanging the parameters $r^{-1}$ and $q$ in the simple components of $I_n$.

A potentially interesting quotient algebra might be the algebra $J_n$. Let $g_i$, $i = 1, 2, \ldots, n-1$ be the images of the braid generators in $J_n$. Then $J_n$ is defined by the relations (besides the braid relations)

\begin{align*}
(R1) \quad & (g_i - q)(g_i + q^{-1})(g_i - r^{-1}) = 0, \\
(R2) \quad & p_i^{(3)} g_i+1 p_i^{(3)}(r^{-1}) = q^{-2} p_i^{(3)} g_i+1 p_i^{(3)}(r^{-1}),
\end{align*}

where $p_i^{(3)}$ is the eigenprojection of $g_i$ for the eigenvalue $\lambda$. Observe that these relations are symmetric with respect to interchanging $q$ with $r^{-1}$. One can show that only using relation (R1) gives us a 24-dimensional quotient of $\mathbb{Q}(r, q) B_3$ as follows: One first checks that words of length $\leq 3$ in $g_i^{\pm 1}$, $g_j^{\pm 1}$ together with $g_i g_j^{-1} g_i g_j^{-1}$ span $\mathbb{Q}(r, q) B_3$ mod (R1); from this one deduces that the dimension of the quotient is at most 24. On the other hand, it is easy to construct sufficiently many nonequivalent irreducible representations (3 1-dimensional, 3 2-dimensional and 1 3-dimensional representation) to exhaust this dimension. This is also in compliance with the fact that the additional relation $\sigma_3^2 = 1$ generates a finite group of order 24 (see [5]). Now it only remains to observe that relation (R2) kills the 2-dimensional representation in which the generators have eigenvalues $r^{-1}$ and $q$. Hence $J_3$ has dimension 20, while $I_3$ only has dimension 19. It is also possible to compute the dimension of $J_4$ in a similar way (for this computer calculations by E. Rowell were needed). It would be interesting to know for which $n$ the algebra $J_n$ has finite dimension.

Remark 5.12. Another possible application of the algebras $I_n$ would be to generalize the approach in [15] for classifying (braided) tensor categories whose Grothendieck semiring is the one of a Lie group. This was successfully carried out in [15] for Lie type $A$, using Hecke
algebra representations of braid groups. Observe that by the uniqueness statement in Theorem 5.4 the braid representations are uniquely determined by the restriction rule, which basically coincides with the tensor product rules of our category (see [34] for further details).

5.4. **Vogel-Deligne approach.** In this approach (see [30], [6] and [7]), the authors consider the decomposition of tensor powers of the adjoint representation of an exceptional Lie algebra as well as of several low rank classical Lie algebras. They observed a number of intriguing similarities in the decomposition behaviour. Further computational evidence for such patterns was given by A. Cohen and R. de Man [4].

In the paper [28], the rigidity of braid representations was used to show that the dimension formulas for representations in the second tensor power of the adjoint representation (given for the Lie group case in [6]) are forced by the Grothendieck semiring and the braiding structure. Here we sketch how the approach in this paper could be applied to tensor powers of adjoint representations, and we give one modest application of it:

As in our setting, one observes that there are only 3 new representations in the second tensor power of the adjoint representation. Hence, switching to the quantum group setting, and restricting to the new part, we again get representations of braid groups whose generators satisfy a cubic equation; for type $E_8$ this coincides with the representations we have studied here. For other types, however, we get more representations than the ones obtained via specializations from $I_n$. So clearly, if there exists an exceptional series, one would expect a larger quotient of the braid groups than the one given by $I_n$. It is easy to check that this quotient would have to be a quotient of the algebra $J_n$, defined at the end of the last section. In fact, it can be shown that the commutants of the action of an exceptional Lie group $g$ on $g_{	ext{new}}^{\otimes n}$, as described in [4], are obtained as classical limits $q \to 1$ of quotients of the algebras $J_n$ for $n \leq 4$; here the eigenvalues of the braid generators are powers of $q$ which can be computed using the formulas for the quantum Casimir (see Prop. 1.6).

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