EXERCISES MATH 202B - 2nd Assignment

1. Let \( G = \mathbb{Z}/3 \), and let \( V \) be a \( \mathbb{C}G \)-module. You can assume that \( V \) is a direct sum \( \bigoplus_j L_j \) of simple \( G \)-modules, and there are only three different simple \( G \)-modules up to isomorphism, \( V_0, V_1 \) and \( V_2 \). The action of \( \overline{1} \) on \( V_i \) is given by multiplication by \( \theta^k \), where \( \theta = e^{2\pi i / 3} \). We would like to determine the number \( m_i \) of simple \( G \)-modules \( L_j \) in the decomposition above which are isomorphic to \( V_i \), for each \( i \).

(a) Let \( \chi_a = Tr(\rho(\overline{a})) \), the trace of the linear map \( \rho(\overline{a}) \) via which \( \overline{a} \) acts on \( V \). Calculate the numbers \( m_i \) in terms of the numbers \( \chi_a \) (hint: use that you can diagonalize the matrices \( \rho(\overline{a}) \)).
(b) Let \( V = \text{span} \{ e_1, e_2 \} \) and define the action of \( \overline{1} \) by \( \overline{1}.e_1 = e_2 \) and \( \overline{1}.e_2 = -e_1 - e_2 \). Use (a) to find the numbers \( m_i \) for \( V \).
(c) Prove in general: Whenever \( V \) is a \( G \)-module where the action of \( \overline{1} \) is represented by a real matrix, then \( m_1 = m_2 \).

2. Let \( F = \mathbb{R} \), and let \( A \) be the subalgebra of \( M_2(\mathbb{R}) \) generated by the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
\end{pmatrix}
\]

(This means \( A \) consists of all polynomials in \( i \)).
(a) Show that \( V = \mathbb{R}^2 \) is a simple \( A \)-module.
(b) Calculate \( \text{End}_A V \).
(c) Consider the same algebra now with \( F = \mathbb{C} \) as a subalgebra of \( M_2(\mathbb{C}) \). What is \( \text{End}_A V \) now? Is it still simple?

3. Let \( V, W \) be \( \mathbb{C}G \)-modules, where \( G \) a finite groups with \( |G| \) elements. Moreover, let \( f : V \rightarrow W \) be a linear map and define \( f(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1}.f(g.v) \).

(a) Show that \( \overline{f} \in \text{Hom}_{\mathbb{C}G}(V, W) \).
(b) Now let \( \phi : V \rightarrow \mathbb{C} \) be any linear map, and let \( V, W \) be simple \( G \)-modules with \( V \not\cong W \). Show that for any \( v \in V \), \( w \in W \) we have \( \sum_{g \in G} \phi(g.v)g^{-1}.w = 0 \). (Hint: Consider the linear map \( f : V \rightarrow W : \overline{v} \mapsto \phi(\overline{v})w \))
(c) Let \( \psi : W \rightarrow \mathbb{C} \) be linear. Show that \( \sum_{g \in G} \phi(g.v)\psi(g^{-1}.w) = 0 \).
(d) Let \( \{ v_1, v_2, \ldots, v_d \} \) and \( \{ w_1, w_2, \ldots, w_e \} \) be bases for \( V \) and \( W \), and let \( \rho_V(g) \) and \( \rho_W(g) \) be the matrices which describe the action of \( g \) on \( V \) and \( W \) with respect to these bases. Prove the following statement, which is part of a result usually referred to as orthogonality relations for matrix coefficients:

\[
\sum_g \rho_V(g)_{ij} \rho_W(g^{-1})_{rs} = 0 \quad \text{for any } 1 \leq i, j \leq d, 1 \leq r, s \leq e.
\]

Hint: Find suitable functionals and vectors to reduce the formula in (c) to the one in (b).
(e) Check the orthogonality relation explicitly for the one-dimensional representations \( V_\theta \) of \( \mathbb{Z}/N\mathbb{Z} \), where \( \theta \) is an \( N \)-th root of unity, and the action is given by \( \overline{1}.v = \theta v \).