EXERCISES MATH 202B - 2nd Assignment

1. Let $G = \mathbf{Z}/3$, and let $V$ be a $\mathbf{C}G$-module. You can assume that $V$ is a direct sum $\bigoplus_j L_j$ of simple $G$-modules, and there are only three different simple $G$-modules up to isomorphism, $V_o, V_1$ and $V_2$. The action of $\bar{1}$ on $V_i$ is given by multiplication by $\theta^i$, where $\theta = e^{2\pi i/3}$. We would like to determine the number $m_i$ of simple $G$-modules $L_j$ in the decomposition above which are isomorphic to $V_i$, for each $i$.

(a) Let $\chi_a = Tr(\rho(\bar{a}))$, the trace of the linear map $\rho(\bar{a})$ via which $\bar{a}$ acts on $V$. Calculate the numbers $m_i$ in terms of the numbers $\chi_a$ (hint: use that you can diagonalize the matrices $\rho(\bar{a})$).

(b) Let $V = \text{span}\ \{e_1, e_2\}$ and define the action of $\bar{1}$ by $\bar{1}.e_1 = e_2$ and $\bar{1}.e_2 = -e_1 - e_2$. Use (a) to find the numbers $m_i$ for $V$.

(c) Prove in general: Whenever $V$ is a $G$-module where the action of $\bar{1}$ is represented by a real matrix, then $m_1 = m_2$.

2. Let $F = \mathbf{R}$, and let $A$ be the subalgebra of $M_2(\mathbf{R})$ generated by the matrix

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(This means $A$ consists of all polynomials in $i$).

(a) Show that $V = \mathbf{R}^2$ is a simple $A$-module.

(b) Calculate $\text{End}_A V$.

(c) Consider the same algebra now with $F = \mathbf{C}$ as a subalgebra of $M_2(\mathbf{C})$. What is $\text{End}_A V$ now? Is it still simple?

3. Let $V$, $W$ be $\mathbf{C}G$-modules, where $G$ a finite groups with $|G|$ elements. Moreover, let $f : V \to W$ be a linear map and define $\tilde{f}(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1}.f(g.v)$.

(a) Show that $\tilde{f} \in \text{Hom}_{\mathbf{C}G}(V, W)$.

(b) Now let $\phi : V \to \mathbf{C}$ be any linear map, and let $V$, $W$ be *simple* $G$-modules with $V \not\cong W$. Show that for any $v \in V$, $w \in W$ we have $\sum_{g \in G} \phi(g.v) g^{-1}.w = 0$. (*Hint*: Consider the linear map $f : V \to W$, $\tilde{v} \in V \mapsto \phi(\tilde{v})w$)

(c) Let $\psi : W \to \mathbf{C}$ be linear. Show that $\sum_g \phi(g.v)\psi(g^{-1}.w) = 0$.

(d) Let $\{v_1, v_2, \ \dots\ v_d\}$ and $\{w_1, w_2, \ \dots\ w_e\}$ be bases for $V$ and $W$, and let $\rho_V(g)$ and $\rho_W(g)$ be the matrices which describe the action of $g$ on $V$ and $W$ with respect to these bases. Prove the following statement, which is part of a result usually referred to as *orthogonality relations for matrix coefficients*:

$$\sum_g \rho_V(g)_{ij} \rho_W(g^{-1})_{rs} = 0 \quad \text{for any } 1 \leq i, j, \leq d,\ 1 \leq r, s \leq e.$$

*Hint* : Find suitable functionals and vectors to reduce the formula in (c) to the one in (b).

(e) Check the orthogonality relation explicitly for the one-dimensional representations $V_\theta$ of $\mathbf{Z}/N\mathbf{Z}$, where $\theta$ is an $N$-th root of unity, and the action is given by $\bar{1}.v = \theta v$.