EXERCISES MATH 202B - 3rd Assignment

1. Let $V$ be a vector space, and let $W = V^m = V \oplus V \oplus \ldots \oplus V$ ($m$ summands). Moreover, let $\psi_j : v \in V \mapsto (0, 0, \ldots, v, \ldots 0)$ ($v$ in the $j$-th slot) and let $\pi_i : (v_j)_j \in V^m \mapsto v_i \in V$. Show:
   (a) Let $b \in \text{End}(W)$ and let $b_{ij} = \pi_i \circ b \circ \psi_j$. Show that the map $b \in \text{End}(W) \mapsto (b_{ij}) \in M_m(\text{End}(V))$ is an isomorphism of algebras.
   (b) Use (a) to show that $\text{End}_A(W) \cong M_m(\text{End}_A(V))$, with notations as in (a) for the $A$-modules $V$ and $W$, where $A$ is an algebra.

2. Let $A$ be the algebra of $n \times n$ matrices over some field $F$ and let $A$ be an $n \times n$ matrix, and let $W = AA$ be the $A$-module (or $A$-left ideal) generated by $A$.
   (a) What are the possible dimensions for $W$? Give an example of $A$ for each such dimension.
   (b) Determine the dimension of $W$ in terms of some numerical invariant associated to a matrix $A$ (such as e.g. the trace, determinant, nullity etc). Obviously not all and possibly none of the given examples may be correct.

3. Let $A = \mathbb{CZ}/3$ and let $W$ be an $A$-module such that the matrices representing the group action all have real coefficients. Determine $\text{End}_A W$ if $Tr(\overline{0}) = 8$ and $Tr(\overline{1}) = -1$.

4. Let $A$ be the algebra of all upper triangular $2 \times 2$ matrices over a field $F$. Consider the $A$-module $V = F^2$, with the usual module action given by matrix multiplication. Compute $B = \text{End}_A(V)$ and $\text{End}_B(V)$. Why is your result compatible with the von Neumann-Jacobson density theorem?

5. Let $V$, $W$ be vector spaces, and let $a \in \text{End}(V)$, $b \in \text{End}(W)$. Define $i(a, b) \in \text{End}(V \otimes W)$ by $i(a, b)(v \otimes w) = a(v) \otimes b(w)$
   (a) Check that indeed $i(a, b)$ is a linear map.
   (b) Show that the map $(a, b) \in \text{End}(V) \times \text{End}(W) \rightarrow i(a, b) \in \text{End}(V \otimes W)$ is bilinear.
   (c) Show that the map $i$ above induces an isomorphism between $\text{End}(V) \otimes \text{End}(W)$ and $\text{End}(V \otimes W)$. 